Mathematics Framework
Chapter 3: Number Sense

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Introduction

The Common Core State Standards are based on an understanding of how young people typically develop mathematical knowledge and skills in a sequenced and cumulative way over time. Knowing about these common learning progressions allows teachers to think about where individual students are in their learning process and what may be most useful to teach next, as well as how the class is progressing as a whole.

Chapters 6, 7, and 8 of this framework describe how teachers at the elementary, middle, and high school levels can use investigations and connections to teach the mathematical big ideas of each grade level. Within this approach, it's important to be able to see how the progression of concepts occurs across all grades, transitional kindergarten through grade twelve (TK–12).[1] This chapter discusses how number sense is embedded within each grade level's big ideas. Moreover, it shows that number sense can itself be described as a progression of big ideas. Those include, for example, in transitional kindergarten through grade two, organize and count with numbers; grades three through five, extend flexibility with numbers; grades six through eight, number line understanding; and grades nine through twelve, seeing parallels between numbers and functions. The chapter emphasizes the growth of number fluency—the ability to use strategies that are flexible, efficient, and accurate—and highlights the value of math talks and games, which encourage students' mental problem solving and communication as well as playful exploration and skill practice; students learn through fun activities while building a positive regard for mathematics. Similarly, chapter 4

[1] For more detailed progressions associated with other concepts, educators may want to consult the Progressions for the Common Core State Standards documents (Common Core Standards Writing Team, 2022), which describe how students develop mathematical understanding from kindergarten through twelfth grade. For a set of detailed progressions for middle grades (four through nine), see the skill map in Teach To One, 2021.

describes how the Standards for Mathematical Practice can be instilled across grade levels, and chapter 5 describes progressions for.

When reading chapters 3, 4, and 5, it's important to keep in mind that although the progressions are described in terms of grade levels, not all students will have mastered the same concepts at exactly the same time and that, wherever they are in their understanding, they should have opportunities to progress in a deliberate manner so as to lay a solid foundation for future learning, rather than to skip over important concepts in order to be instructed only on grade-level standards. Students often hold sophisticated understandings in some domains while needing to solidify more basic understandings in others. The progressions should help teachers identify and fill gaps in understanding where they appear, as well as to identify next steps to take to support advancement. Teaching with an appreciation for how ideas build on each other can often help students accelerate their learning.

Development of Students' Number Sense Across the Grades

From the time children can talk, and possibly even before, their relationship to the world is imbued with an understanding of numbers. Before any formal instruction begins, a child's understanding of numbers and the role that numbers play in life originates from a place of context. As they start to explore, children use numbers as a way to help describe what they see and to gauge their own place in the world. Describing their age, for example, is often one of the first ways children use numbers, and they see their age number growing and changing as they do. When one child asks another, "How many are you?" the question seeks to utilize a numeric response to gain insight into others and themselves, since they know that age indicates experience, growth, access to privileges, and so on. Children may hold up fingers to represent their own age, or count by rote, "1, 2, 3, …." to describe how many pets, toys, or cookies they see.

Children continue to use numbers when at play or engaged in daily activities. In transitional kindergarten (TK), students count as they play games, sing, or help with classroom tasks. Elementary-age children make comparisons (who has more?), keep score, and tell and track time. As preteens, they pursue more personal and social interests, and numbers play a role in helping them make decisions about saving and spending money, scheduling time with friends, and managing their free time. Extracurricular activities such as music, athletics, or video games also present situations that call for numerical thinking. Such number-related interests grow in sophistication as students transition to the teenage years. As adolescents start to gain a measure of independence, they rely on numbers that inform their decisions about budget, shopping, and saving for future endeavors. Adults use numbers on a day-to-day basis for cooking, shopping, keeping track of household finances, and while engaging in community activities such as fundraising and civic engagement. Thus, a strong foundation in the use and understanding of numbers, developed throughout the school years, is critical in preparing young community members to continue to make sense of the world and to make wise decisions as adults.

Number sense is multifaceted, and while components can be recognized easily, the concept is difficult to define. The operating definition of *number sense* for this chapter is a form of intuition that students develop about number (or quantity). As students increase their number sense, they can see relationships between numbers readily, think flexibly about numbers, and notice patterns that emerge as they work with numbers. Students who have developed number sense think about numbers holistically rather than as separate digits and can devise and apply procedures to solve problems based on the particular numbers involved. Summarily, “number sense reflects a deep understanding of mathematics, but it comes about through a mathematical mindset that is focused on making sense of numbers and quantities” (Boaler, 2016).

While students enter school possessing varying levels of number sense, research shows that this knowledge is not an inherited capacity. Instead, “number sense is something that can be improved, although not necessarily by direct teaching. Moving between representations and playing games can help children’s number sense development” (Feikes and Schwingendorf, 2008). By deemphasizing the reliance on memorized facts and instead encouraging flexibility in thinking about numbers, such as seeing multiple ways to compose and decompose numbers and quantities, teachers can help support all students in accessing more sophisticated strategies. The acquisition of a rich, comfortable number sense is incremental and is enriched by play, both inside and outside the classroom. When educators encourage, recognize, and value students’ emerging number sense, they support students’ growth as mathematically capable, independent problem solvers.

Instruction that relies on the principles of mathematics and precise mathematical language strengthens number sense and minimizes the development of lasting misconceptions. From the youngest grades, the mathematical language used in classrooms needs to be accurate so that students are prepared for the mathematics they will learn in subsequent grades. Primary grade students, for example, may hear some version of “you can’t take a bigger number from a smaller number,” which is only the case for the set of whole numbers. This can lead to genuine confusion when students later encounter operations with integers. In the online resource Nix the Tricks (Cardone, 2015) and the article “13 Rules That Expire” (Karp et al., 2014), the authors advise that by avoiding teaching “tricks” and short-lived rules, teachers can do much to help students learn “real” mathematics as big ideas that are related to one another rather than a list of procedures and tricks that must be memorized.

Corollary to precise mathematical language is the critical need to support mathematics learning through literacy and language development. Instruction for linguistically and culturally diverse English learners who are developing mathematical proficiency should be rooted in and informed by the California English Language Development Standards (CA ELD Standards). The first stated purpose of the CA ELD Standards is to establish expectations of the knowledge and familiarity with English necessary in various contexts for diverse English learners. Knowledge of and alignment with the CA ELD Standards offers mathematics educators ways to strengthen instructional support that benefits all students. Building comprehensive mathematics instruction based on an understanding of individual CA ELD Standards ensures that learning reflects a meaningful and relevant use of language that is appropriate to grade level, content area, topic, purpose, audience, and text type.

Instruction in the elementary grades should provide students with frequent, varied, culturally relevant, interesting experiences to promote the development of number sense. These experiences need to include sustained investigations in which children explore numerical situations for an extended time in order to initiate, refine, and deepen their understanding. Students further strengthen their number sense when they communicate ideas, explain reasoning, and consider the reasoning of others. Such experiences give each student the opportunity to internalize a cohesive structure for numbers that is both robust and consistent. The eight California Common Core Standards for Mathematical Practice (SMP), implemented in tandem with the California Common Core Content Standards for Mathematics, offer a carefully constructed pathway that supports the gradual growth of number sense across grade levels.

The Content Connections (CCs, initially presented in chapter one) organize mathematical content and connect the big ideas that span TK–12 in this framework. Two of the CCs are particularly associated with number sense. In working with numbers, students develop an understanding of how numbers measure quantities and their change and how numbers can fit together or be taken apart. The CCs most applicable to this chapter are CC2, Exploring Changing Quantities, and CC3, Taking Wholes Apart and Putting Parts Together. CC2 and CC3 are mentioned throughout this chapter when they apply. In addition, CC1, Reasoning with Data, and CC4, Discovering Shape and Space, play a prominent role at times in developing number sense. For example, CC1 and CC4 apply when students measure attributes of objects and categorize numbers of objects.

This chapter presents a progression of activities and tasks aligned with standards and organized by grade bands (TK–2, 3–5, 6–8, and 9–12), demonstrating how number sense underlies much of the mathematics content that students encounter across the school years. Each grade-band section identifies big ideas that connect across grades (see figure 3.1). These ideas can provide guidance for teachers as they seek to develop their students' robust understanding of numbers and help them maintain focus on important learning.

Figure 3.1 Big Ideas to Be Presented in Each Grade-Level Band

TK–2	3–5	6–8	9–12
• Organize and count with numbers • Compare and order numbers • Learn to add and subtract, using numbers flexibly	• Extend flexibility with number • Understand the operations of multiplication and division • Make sense of operations with fractions and decimals • Use number lines as tools	• Demonstrate number line understanding • Develop an understanding of ratios, percents, and proportional relationships • See generalized numbers as leading to algebra	• See parallels between numbers and functions • Develop an understanding of real and complex number systems • Develop financial literacy

The grade-band chapters include sample number-related questions and tasks representative of each grade. These illustrate how students can use number sense across the grades to meet the expectations in the SMPs and the content standards in the California Common Core State Standards for Mathematics (CA CCSSM). Because math talks, particularly number talks and number strings, and games are especially powerful means of cultivating number sense, a Math Talks section is included for each grade band (see page 23 for transitional kindergarten through grade five and page 54 for grades six through twelve in this chapter). Fluency in mathematics is defined and described in the box below, since it is a topic is of continuing importance across all grade levels.

Fluency

Fluency is an important component of mathematics; it contributes to a student’s success through the school years and remains useful in daily life as an adult. What is meant by fluency in elementary grade mathematics? Fluency means that students use strategies that are flexible, efficient, and accurate to solve problems in mathematics. For example, content standard 3.OA.7 calls for third-graders to “Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division … or properties of operations.” Students who are comfortable with numbers and who have learned to compose and decompose numbers strategically develop fluency along with conceptual understanding. They can use known facts, including those drawn from memory, to determine unknown facts. They understand, for example, that the product of 4 × 6 will be twice the product of 2 × 6, so that if they know 2 × 6 = 12, then 4 × 6 = 2 × 12, or 24. The more familiar students become with addition, subtraction, multiplication, and division facts, and the more readily they use them, the more able they are to handle complex, multistep problems. In composing and decomposing numbers, students experience a fundamental idea—Content Connection 3 (CC3, Taking Wholes Apart and Putting Parts Together; see chapter one).

In the past, fluency has sometimes been equated with speed, which may account for the common, but counterproductive, use of timed tests for practicing facts (Henry and Brown, 2008). Fluency involves more than speed, however, and requires knowing, efficiently retrieving, and appropriately using facts, procedures, and strategies, including from memory. Achieving fluency builds on a foundation of conceptual understanding, strategic reasoning, and problem solving (NGA Center and CCSSO, 2010; NCTM, 2000, 2014). To develop fluency, students need to have opportunities to explicitly connect their conceptual understanding with facts and procedures (including standard algorithms) in ways that make sense to them (Hiebert and Grouws, 2007).

Primary Grades, Transitional Kindergarten Through Grade Two

In the primary grades, students begin the important work of making sense of the number system, implementing SMP.2 to “Reason abstractly and quantitatively.” Students engage deeply with CC3 (Taking Wholes Apart and Putting Parts Together) as they learn to count and compare, decompose, and recompose numbers. Building on a transitional kindergarten understanding that putting two groups of objects together will make a bigger group (addition), kindergarteners learn to take groups of objects apart, forming smaller groups (subtraction). They develop an understanding of the meaning of addition and subtraction and use the properties of these operations. Young students need frequent experiences using concrete materials to make sense of problems, creating representations of their strategies, and have meaningful discussions about their mathematical thinking. They actively manipulate concrete tools (e.g., fingers, blocks, clocks, tiles, etc.) to develop their understanding of quantity.

Concepts of place value, comparison of numbers, and the ability to use flexible strategies to add and subtract are of premier importance as preparation for the mathematics to follow. In grades three through five, students will apply and extend their place-value understanding to larger numbers, decimals, and fractions. They will develop understanding of multiplication and division, refine strategies for computation for all four arithmetic operations, and begin to use some standard algorithms. (The use of mathematical tools to support sense-making is emphasized throughout *Chapter 6, Investigating and Connecting, Transitional Kindergarten through Grade Five*.)

Figure 3.2 shows how students' number sense foundation begins with quantities encountered in daily life before progressing to more formal work with operations and place value.

Figure 3.2: TK–2 Alignment Between the California Preschool Learning Foundations and the California Common Core State Standards for Mathematics (Kindergarten)

California Preschool Learning Foundations Mathematics	**California Common Core State Standards for Kindergarten Mathematics**
Number Sense	Counting and **Cardinality**
Children understand numbers and quantities in their everyday environment.	• Know number names and the count sequence. • Count to tell the number of objects. • Compare numbers.
Children understand number relationships and operations in their everyday environment	Operations and Algebraic Thinking • Understand addition as putting together and adding to, and subtraction as taking apart and taking from Number and Operations in Base Ten • Work with numbers 11–19 to gain foundations for place value

Source: California Department of Education, 2015a, 37.

Three big ideas (Boaler, Munson, and Williams, 2018) related to number sense for transitional kindergarten through grade two call for students to do the following:

- Organize and count with numbers
- Compare and order numbers
- Operate with numbers flexibly

Students who acquire number sense in these grades use numbers comfortably and intentionally to solve mathematical problems. They select or invent sensible calculation strategies to make sense in a particular situation, developing as mathematical thinkers. All students, including students who are English learners and those with learning differences, benefit from instruction that allows for peer interaction and support, multiple approaches, and multiple means of representing their thinking. (See chapter two for principles of the Universal Design for Learning and strategies for English language development.)

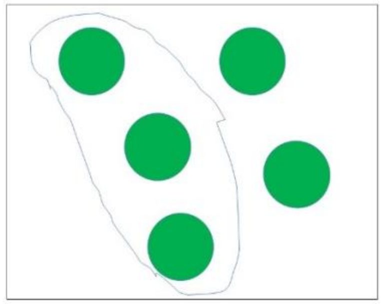
How Do Students in Transitional Kindergarten Through Grade Two Organize and Count Numbers?

Transitional Kindergarten

The work of learning to count typically begins in the preschool years. Often, young children who have not yet developed a mental construct of the quantity "10" can recite the numbers 1 to 10 fluently. In transitional kindergarten, children learn to count objects meaningfully: They touch objects one by one as they name the quantities, and they recognize that the total quantity is identified by the name of the last object counted (SMP.2, 5; PLF.NS–1.4, 1.5).

Kindergarten

In kindergarten, children become familiar with numbers from 1 to 20 (K.CC.5). They count quantities up through 10 accurately when presented in various configurations. Dot pictures can be an effective tool for developing counting strategies. With practice, students learn to subitize (recognize a quantity without needing to count) small quantities, 1 to 5. Presented with quick images of small amounts such as 1, 2, and 3, children use what they can perceive innately as subitized units to compose and decompose larger amounts, such as 5 and then 10, as they work to develop more productive strategies than counting all and counting on. A child who can subitize 3 can see the image below as 3 + 2, to make a total of 5.



Counting Collections is a structured activity in which students work with a partner to count a collection of small objects and make a representation of how they counted the collection (Franke et al., 2018; Schwerdtfeger and Chan, 2007). While students count, the teacher circulates to observe progress, noting and highlighting counting strategies as they emerge.

Standard K.OA.3 calls for students to decompose numbers up to 10 into pairs in more than one way and to record each decomposition by a drawing or an equation. As examples of CC3, students may use counters to build the quantity 5 and discover that 5 = 5 + 0, 5 = 4 + 1, 5 = 3 + 2, 5 = 2 + 3, 5 = 1 + 4, and 5 = 0 + 5. Such explorations give students the opportunity to see patterns in the movement of the counters and connect that observation to patterns in the written recording of their equations. As they engage in number sense explorations, activities, and games, students develop the capacity to reason abstractly and quantitatively (SMP.2) and to model mathematical situations symbolically and with words (SMP.4).

Grade One

First-grade students undertake direct study of the place-value system. They compare two two-digit numbers based on the meanings of the tens and ones digits, a pivotal and somewhat sophisticated concept (SMP.1, 2; 1.NBT.3). To gain this understanding, students need to have worked extensively creating tens from collections of ones and to have internalized the idea of a “10.” Students may count 43 objects, for example, using various approaches. Younger learners typically count by ones and may show little or no grouping or organization of 43 objects as they count. As they acquire greater confidence and skill, children can progress to counting some of the objects in groups of 5 or 10 and perhaps will still count some objects singly.

Once the relationship between ones and tens is better understood, students tend to count the objects in a more adult fashion (SMP.7), grouping objects by tens as far as possible (e.g., four groups of 10 and three units). Teachers support student learning by providing interesting, varied, and frequent counting opportunities using games, group activities, and a variety of tools along with focused mathematical discourse. Choral Counting is fun for students and can be a powerful means of encouraging pattern discovery, reasoning about numbers, and problem solving. An effective Choral Counting experience includes a public recording of the numbers in the sequence (e.g., counting by threes starting with 4: 4, 7, 10, 13, 16…) and a discussion in which students share their reasoning as the teacher helps students extend and connect their ideas (Chan Turrou et al., 2017).

Posing questions as students are engaged in the activities can help children see relationships and further develop place-value concepts. A technique described as "Notice and Wonder" can be an effective means of increasing student understanding as well as involvement when faced with a problem-solving challenge. By inviting students to express anything they notice in a problem, teachers create a safe environment. Students share their thoughts without any pressure to answer or solve a problem.

Some questions in the instance of counting 43 objects might be:

- What do you notice?
- What do you wonder?
- What will happen if we count these by singles?
- What if we counted them in groups of tens?
- How can we be sure there really are 43 here?
- I see you counted by groups of tens and ones. What if you counted them all by ones? How many would we get?

While the impulse may be to tell students that the results will be the same with either counting method, direct instruction is unlikely to make sense to them at this stage. Children must construct this knowledge themselves (Van de Walle et al., 2014).

Grade Two

Students in second grade learn to understand place value for three-digit numbers. They continue the work of comparing quantities with meaning (2.NBT.1) and record these comparisons using the <, =, and > symbols. They engage in CC3 when they recognize 100 as a “bundle” of 10 tens and use that understanding to make sense of larger numbers of hundreds (200, 300, 400, etc.) up to 1000 (SMP.6, 7). For numbers up to 1000, they use numerals, number names, and expanded form as ways of expressing quantities.

Examples:

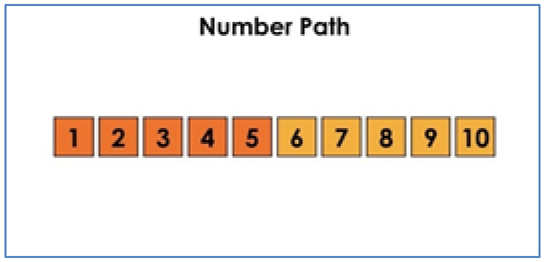
- To solve 18 + 7, a child may think of 7 as 2 + 5, so 18 + 7 = 18 + 2 + 5 = 20 + 5, which is easier to solve
- 234 = 200 + 30 + 4; 243 = 200 + 40 + 3. Then, 234 < 243.

Second-grade students who have developed an understanding of place value for three-digit numbers are building a foundation for later grades in which they will work with large whole numbers and decimals.

How Do Students in Transitional Kindergarten Through Grade Two Learn to Compare and Order Numbers?

Transitional Kindergarten

With extensive practice of counting, transitional kindergarten students establish the foundation for comparing numbers, which later enables them to locate numbers on a number path (shown below) and in grade two, on a number line.



Transitional kindergarten students engage in activities that introduce the relational vocabulary of *more, fewer, less, same as, greater than, less than,* and *more than.* These activities should be designed in ways that provide students with a variety of structures to practice, engage in, and eventually master the vocabulary. Effective instructors model these behaviors, provide explicit examples, and share their thought process as they use the language. In Tier 1 (core) instruction, teachers can create rich, effective discussion where students use developing skills to clarify, inform, question, and eventually employ these conversational behaviors without direct prompting (Shapiro, n.d.). Such intention supports all students, including linguistically and culturally diverse English learners, and ensures all learners develop both mathematics content and language facility. Children compare two collections of small objects as they play fair share games and decide "Who has more?" By lining up the two collections side by side, children can make sense of the question and practice the relevant vocabulary. They investigate the sequence of numbers on a 0 to 99 or 1 to 100 chart or build a number path to order numbers. As students develop skills in recognizing numerals (PLF.NS–1.2), they can play games with cards, such as Compare (comparing numerals or sets of icons on cards). Each student receives a set of cards with numerals or sets of objects on them (within five). Working with a partner, each student flips over one card (like the card game War). The students decide which card represents *more* or *fewer* or whether the cards are the *same as* (PLF.NS–2.1; SMP.2; adapted from *2013 Mathematics Framework*, 43)

Kindergarten

Students continue to identify whether the number of objects in one group is greater than, less than, or equal to the number in another group (K.CC.6) by building small groups of objects and either counting or matching elements within the groups to compare quantities. They learn to add to a group of objects and learn that when an additional item is added, the total number increases by one. Students may need to recount the whole set of objects from one, but the goal is for students to count on from an existing number of objects. This is a conceptual start for the grade-one skill of counting up to 120 starting from any number. Children need considerable repetition and practice with objects they can touch and move to gain this level of abstract and quantitative reasoning (MP.2, 5).

Grade One

The concept that a 10 can be thought of as a bundle of 10 ones—called a “10” (1.NBT.2a)—is developed in first grade. Students must understand that a digit in the tens place has greater value than the same digit in the ones place (i.e., four tens is greater than four ones) and apply this understanding to compare two two-digit numbers and record these comparisons symbolically (1.NBT.3). Students use quantitative and abstract reasoning to make these comparisons (SMP.2) and examine the structure of the place-value system (SMP.7) as they develop these essential number concepts. Teachers can have students assemble bundles of 10 objects (popsicle sticks or straws, for example) or snap together linking cubes to make tens as a means of developing the concept and noting how the quantities are related. Repetition and guided discussions are needed to support deep understanding.

Grade Two

In second grade, students extend their understanding of place value and number comparison to include three-digit numbers. This learning must build upon a strong foundation in place value at the earlier grades. To compare two three-digit numbers, second-graders can take the number apart by place value and compare the number of hundreds, tens, and ones, or they may use counting strategies (SMP.7; 2.NBT.4). For example, to compare 265 and 283, the student can view the numbers as 200 + 60 + 5 compared with 200 + 80 + 3, and note that while both numbers have two hundreds, 265 has only six tens, while there are eight tens in 283, so 265 < 283. Another strategy relies on counting: a student who starts at 265 and counts up until they reach 283 can observe that since 283 came after 265, 265 < 283 (MP.7). Grade two students, who have been using number paths (see chapter six) in earlier grades, are now positioned to order numbers on a number line. Students who have made sense of comparing and ordering whole numbers will be able to use that understanding as they encounter larger numbers, fractions, and decimal values in grades three through five.

How Do Students Learn to Add and Subtract Using Numbers Flexibly in Transitional Kindergarten Through Grade Two?

Students develop meanings for addition and subtraction as they encounter problem situations in transitional kindergarten through grade two. Addition situations involve combining or adding to quantities; subtraction situations include taking groups apart, taking from, comparing, and finding the difference between two quantities. (See also the table “Common Addition and Subtraction Situations” in chapter six.) Note that subtraction sometimes, but not always, involves the action of “taking away,” and therefore the terms “subtract” and “take away” are not synonymous. Depending on the problem context, a subtraction problem may be understood and represented as a comparison situation or a question about the difference between two quantities, which does not indicate that anything is taken away. It is important that precise language be associated with subtraction from these early grades to avoid misconceptions that interfere with learning in later mathematics.

As they progress through transitional kindergarten through grade two, students expand their ability to represent problems, and they use increasingly sophisticated computation methods to find answers. The quality of the situations, representations, and solution methods selected significantly affects growth from one grade to the next.

Transitional Kindergarten

Young learners acquire facility with addition and subtraction while using their fingers, small objects, and drawings during purposefully designed “play.” They engage in activities that require thinking about and showing one more or one less, and they put together or take apart small groups of objects. When two children combine their collections of blocks or other counting tools, they discover that one set of three added to another set of four makes a total of seven objects. At the transitional kindergarten level, the total is typically found by recounting all seven objects (PLF.NS–2.4). Students need frequent opportunities to act out and solve story situations that call for them to count, recount, put together, and take apart collections of objects in order to develop understanding of the operations. Exercises such as having students compose their own addition and subtraction stories for classmates to consider empower young learners to view themselves as thinkers and doers of mathematics (SMP.3, 4).

Kindergarten

Kindergarteners develop understanding of the operations of addition and subtraction actively and tactilely. They consider “addition as putting together and adding to and subtraction as taking apart and taking from” (K.OA.1–5). Students add and subtract small quantities using their fingers, objects, drawings, and sounds and by acting out situational problems or explaining verbally (K.OA.1). These means of engagement reflect the CA ELD Standards in that they ensure English learners are supported by structures that allow for active contributions to class and group discussions, including scaffolds to ask questions, respond appropriately, and provide meaningful feedback.

As students develop their understanding of addition and subtraction, it is essential that they discuss and explain the ways in which they solve problems so that they are simultaneously embodying key mathematical practices. As teachers invite students to use multiple strategies (SMP.1), they bring attention to various representations (SMP.4) and encourage students to express their own thinking verbally and listen carefully as other students explain their thinking (SMP.3, 6).

Grade One

First-graders use addition and subtraction to solve problems within 100 using strategies and properties such as commutativity, associativity, and identity. Students focus on developing and using efficient, accurate, and generalizable methods, although some students may also use invented strategies that are not generalizable. For example, three children solve 18 + 6:

> Clara: “I just counted up from 18. I did 19, 20, 21, 22, 23, 24.” (generalizable, accurate).

> Malik: “I broke the 6 apart into 2 + 4, and then I added 18 + 2, and that’s 20. Then I had to add on the 4, so it’s 24.” (efficient, flexible, generalizable).
>
> Asha: “I know 6 = 3 + 3, so I added 18 + 3 and that was 21, then 3 more was 24.” (flexible).

In this situation, the teacher may choose to conduct a brief discussion of these methods, inviting students to comment on which method(s) work all the time, which are easiest to understand, or which they might wish to use again for another addition problem. The teacher notes that Malik and Asha naturally used CC3 in their invented strategies. Class discussions that allow students to express and critique their own and others’ reasoning are instrumental in supporting flexible thinking about numbers and the development of generalizable methods for addition and subtraction (SMP.2, 3, 4, 6, 7). Note that while students in first grade do begin to add two-digit numbers, they do so using strategies as distinguished from formal algorithms. The CA CCSSM intentionally place the standard algorithms for addition and subtraction in fourth grade (4.NBT.4).

It is imperative that students implement a standard method only after they have fully developed understanding of the operation, can connect previous strategies and representations to the steps of the algorithm, and make sense of this abstract process. Students who use invented strategies before learning standard algorithms understand base-10 concepts more fully and are better able to apply their understanding in new situations than students who learn standard algorithms first (Carpenter et al., 1997). The Progressions for the Common Core State Standards documents are a rich resource (Common Core Standards Writing Team, 2022); they describe how students develop mathematical understanding from kindergarten through twelfth grade. The Progression on K–5 Number and Operations in Base Ten discusses the distinction between strategies and algorithms, describes variations in standard algorithms that have advantages or disadvantages, and shows how the use of standard algorithms grows out of and relates to understanding and skill with each operation. Further discussion of the role of algorithms in elementary grades is included in chapter six (see the table “Development of Fluency with Standard Algorithms, Elementary Grades”).

Some strategies to help students develop understanding and fluency with addition and subtraction include the use of 10-frames or math drawings, rekenreks, comparison bars, and number-bond diagrams. The use of visuals (e.g., hundreds charts, 0 to 99 charts, and number paths) can also support fluency and number sense.

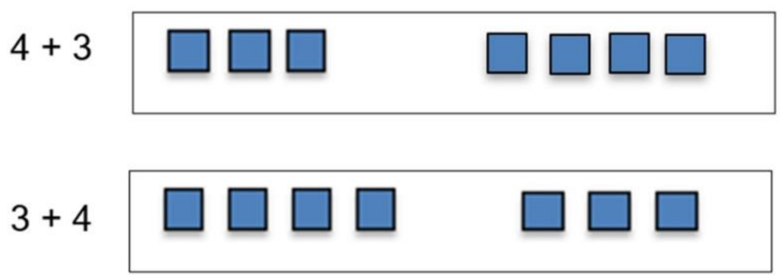
How does a first-grader use properties of operations?

- Commutative Property

  When students use direct modeling in addition situations, they discover that the sum of two numbers is the same despite changing the order of the addends.

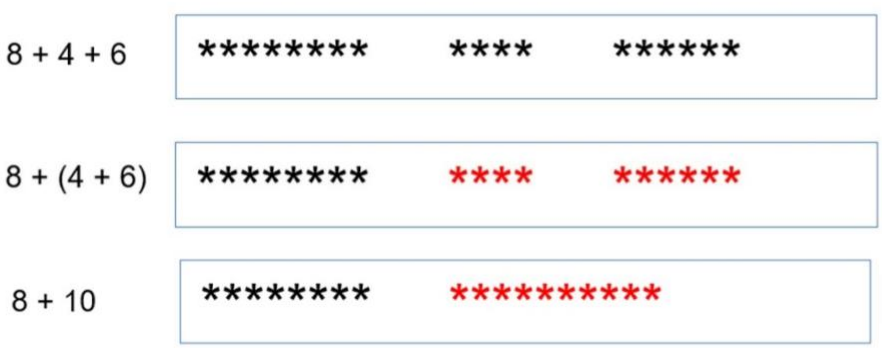
  Example: Using blocks, a child models 3 + 4 and finds the sum is 7.

  Next, the child models 4 + 3 and again finds a sum of 7 and notes that the order in which the numbers were added did not make a difference in the result.



- Associative Property

  To add 8 + 4 + 6, the child “sees” a 10 in 4 + 6, so first adds 4 + 6 = 10, and then adds the 8, and finds that 8 + 10 = 18.



8 + 4 + 6 = 18

****

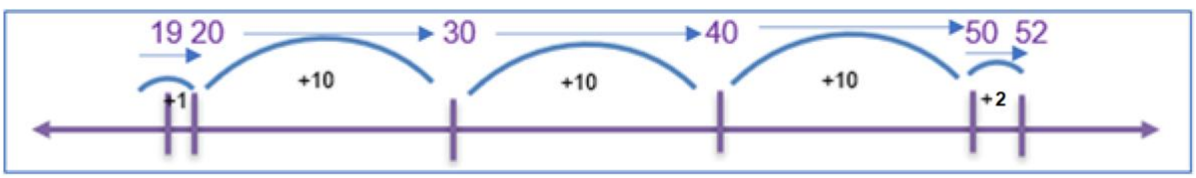
- Identity Property

Asked to solve 8 + 0, the first-grader counts out eight cubes and says, “That’s all because there’s no more cubes to add.”

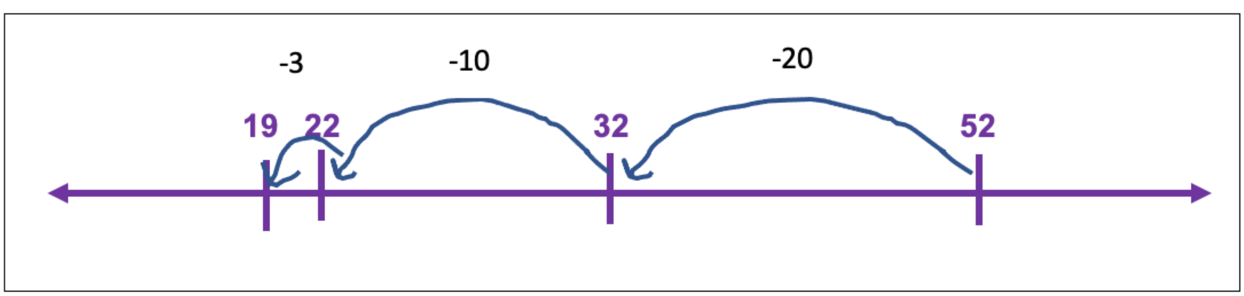
Grade Two

Students in second grade add and subtract numbers within 1000 and explain why addition and subtraction strategies work, using place value and the properties of operations (2.NBT.7, 2.NBT.9, SMP.1, 3, 7). They continue to use concrete models, drawings, and number lines and work to connect their strategies to written methods. Many of the strategies involve taking numbers apart or fitting them together (CC3).

Example: Second-graders use “jumps” on a number line (below) to solve 52 – 19.



Student A: “I started at 19 and went to 20; that was + 1. Then 20 to 30 is 10, and 30 to 40 is 10, and 40 to 50 is 10 more, so that’s 10 + 10 + 10 plus the 1, so that’s 31. And 2 more to get to 52, so it’s 33. 19 + 33 = 52.”



Student B: “I did 52 – 20 = 32, but then I needed to subtract 10 more, so 32 – 10 = 22, and then I’m getting close! 22 – 2 = 20, and I know 20 – 1 = 19. So 20 + 10 + 3 = 33.”

Student C: “Mine was like yours, but a little bit different. I started at 52, too, but I went 52 – 30 = 22, and then I only had to take away 3 more to get down to 19. So it’s 52 – 30 = 22, and 22 – 3 = 19. So there’s 30 + 3 = 33.”

Note that all three children used number sense strategies to solve the problem and were able to explain their thinking. Student A used counting up (addition) to solve 52 – 19, while students B and C subtracted, moving down the number line from 52 to 19.

Second-graders explore many addition and subtraction contextual problem types, including working with result unknown, change unknown, and start unknown problems (California Department of Education, 2015b). Students in transitional kindergarten through grade two who employ mathematical practices (especially SMP.1, 2, 3, 4, 7), along with effective, accurate strategies for calculating in a variety of addition and subtraction situations, will be equipped to understand and make use of standard algorithms in subsequent grades.

Math Talks, Transitional Kindergarten Through Grade Five

Math talks, which include number talks, number strings, and number strategies, are short discussions (typically about 10–15 minutes) in which students solve a mathematics problem mentally, share their strategies aloud, and determine a correct solution as a whole class (SMP.2, 3, 4, 6). Number talks can be viewed as “open” versions of computation problems in that in a number talk, each student is encouraged to invent or apply strategies that will allow them to find a solution mentally and to explain their approach to peers. The notion of using language to convey mathematical understanding aligns with the key components of the CA ELD Standards. The focus of a math talk is on comparing and examining various methods so that students can refine their own approaches, possibly noting and analyzing any error they may have made.

Participation in math talks provides opportunity for learners of English to interact in meaningful ways, as described in the CA ELD Standards (26–7). Effective math talks can advance students' capacity for collaborative, interpretive, and productive communication.

In the course of a math talk, students often adopt methods another student has presented that make sense to them. Math talks designed to highlight a particular type of problem or useful strategy serve to advance the development of efficient, generalizable strategies for the class. These class discussions provide an interesting challenge and a safe situation in which to explore, compare, and develop strategies. While engaged in math talks, students in transitional kindergarten through grade two develop counting strategies, build place-value concepts, work with the operations of addition and subtraction, compare and contrast geometric figures, and more. In grades three through five, math talks help students strengthen, support, and extend their place-value understanding, calculation strategies, and fraction concepts as well as develop geometric concepts. Student drawings are helpful for many of these topics, and math talks can advance learning progressions in important ways.

Several types of math talks are appropriate for transitional kindergarten through grade two, including the following:

- Dot talks: A collection of dots is projected briefly (for just a few seconds), and students explain how many they saw and the method they used for counting the dots.
- Ten-frame pictures: An image of a partially filled 10-frame is projected briefly, and students explain various methods they used to figure out the quantity shown in the 10-frame.
- Calculation problems: Either an addition or subtraction problem is presented, written in horizontal format and involving numbers that are appropriate for the students' current capacity. Presenting problems in horizontal format increases the likelihood that students will think strategically rather than limit their thinking to an algorithmic approach. For example, first-graders might solve 7 + ? = 11 by thinking "7 + 3 = 10, and 1 more makes 11." Second-graders subtract two-digit

numbers. To solve 54 - 25 mentally, they can think about 54 - 20 = 24, and then subtract the 5 ones, finding 24 - 5 = 19.

For grades three through five, possible topics for math talks include the following:

- Multiplication calculations for which students can use known facts and place-value understanding and apply properties to solve a two-digit by one-digit problem. Presenting such calculation problems in horizontal format increases the likelihood that students will use a range of methods.
- Students can use relational thinking to consider whether 42 + 19 is greater than, less than, or equal to 44 + 17, and explain their strategies.
- Asking students to order several fractions mentally or with math drawings encourages the use of strategies such as common numerators and benchmark fractions. For example: arrange in order, from least to greatest, and explain how you know: 4/5, 1/3, 4/8.

Opportunities to explain their own reasoning and listen to and critique the reasoning of others are essential for students to make sense of each problem type. In the vignette *Number Talk with Addition, Grade Two* second-graders use and explain strategies based on place value and properties of operations and several mathematical practices as they solve two-digit addition problems mentally. The vignette also illustrates the value of number talk, in this instance to expand students understanding of taking things apart and refitting them back together.

Games, Transitional Kindergarten Through Grade Five

Games are a powerful means of engaging students in thinking about mathematics. Using games and interactives to replace standard practice exercises contributes to students' understanding as well as their affect toward mathematics (Bay-Williams and Kling, 2014). Games typically engage students in peer-to-peer oral communication and represent another opportunity to engage students' conversation around mathematic vocabulary in a low-stakes environment.

A plethora of rich activities related to number sense topics are offered at Nrich Maths' website (University of Cambridge, n.d.). For example, the Largest Even game allows students to explore combinations of odd and even numbers in a game format, either online or on paper. The game allows for the discovery of informal "rules," such as an odd number plus an odd number is an even number, while an odd number plus an even number yields an odd sum. As they develop winning moves, students practice addition repeatedly and build skill and confidence with the operations as well as deeper understanding of odd and even numbers. The Factors and Multiples game, appropriate for grades three through five, challenges students to find factors and multiples on a hundreds grid in a game format, either online or on paper. As students discover strategies based on prime and square numbers, they develop winning moves and gain insight and confidence in recognizing multiples, primes, and square numbers.

The Math Playground website (Math Playground, n.d.) provides a range of games for practicing skills, logic puzzles, story problems, and some videos intended for students in grades one through eight.

Intermediate Grades, Three Through Five

The upper-elementary grades present new opportunities for developing and extending number sense. Four big ideas related to number sense for grades three through five (Boaler, Munson, and Williams, 2018) call for students to:

- Extend their flexibility with number
- Understand the operations of multiplication and division
- Make sense of operations with fractions and decimals
- Use number lines as tools

Graham Fletcher presents a series of videos that vividly illustrate how key elementary topics are developed across grades three through five. Three videos, *Progression of Multiplication, Progression of Division, and Fractions: The Meaning, Equivalence, & Comparison*, examine particularly pertinent content and are useful resources for teachers of these grades (Gfletchy, n.d.) as well as for parents.

How Is Flexibility with Number Developed in Grades Three Through Five?

Grade Three

A third-grade student's ability to add and subtract numbers to 1000 fluently (3.NBT.2) is largely dependent on their ability to think of numbers flexibly, to compose and decompose numbers (CC3), and to recognize the inverse relationship between addition and subtraction. For example, a third-grader mentally adds 67 + 84, decomposing by place value, and recognizing that 67 + 84 = (60 + 80) + (7 + 4) = 140 + 11 = 151. Another student, noting that 67 is close to 70, adjusts both addends: 67 + 84 = 70 + 81. Choosing to solve the easier problem, the student computes 70 + 81 = 151.

Children who have not yet made sense of numbers in these ways often calculate larger quantities without reflection, sometimes getting unreasonable results. By using number sense, students can note that 195 is close to 200, so they estimate, before calculating, that the difference between 423 and 195 will be a bit more than 223. This kind of thinking can develop only, as noted above, if students have sufficient, sustained opportunities to "play" with numbers, to think about their relative size, and to estimate and reflect on whether their answers make sense (SMP.3, 7, 8). Students who have developed understanding of place value for three-digit numbers and the operation of subtraction may calculate to solve 423 – 195 in a variety of ways.

Note the following examples of students' thinking and recording of calculation strategies:

Student A	Student B
“I subtracted 200, but that’s a little bit too much, so I added back 5.” $$\begin{array}{r} 423 \\ -\ 200 \\ \hline 223 \end{array}$$ 223 + 5 = **228**	“First I subtracted 100, because that’s easy, and that was 323. Then I subtracted 90, and got to 233 and then subtracted 5 more, so it’s 228.” $$\begin{array}{r} 423 \\ -\ 100 \\ \hline 323 \end{array} \quad \begin{array}{r} 323 \\ -\ 90 \\ \hline 233 \end{array} \quad \begin{array}{r} 233 \\ -\ 5 \\ \hline 228 \end{array}$$

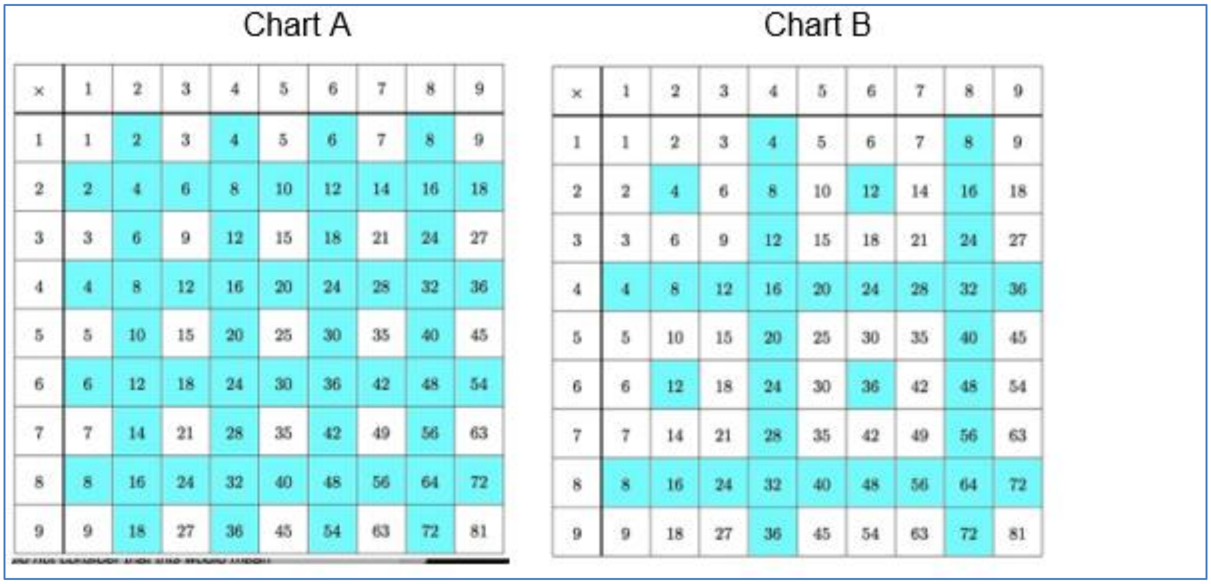
Grade Four

After their introduction to multiplication in third grade, fourth-grade students employ that understanding to identify prime and composite numbers and to recognize that a whole number is a multiple of each of its factors (4.OA.4). An activity such as Identifying Multiples, found at Illustrative Mathematics (Illustrative Mathematics, n.d.a), provides a reflective mathematics experience in a visually interesting format. Students explore the multiplication table and, by highlighting multiples with color, see patterns and relationships. This visual approach serves to cultivate and expand number sense as well as to provide access for linguistically and culturally diverse English learners and to those for whom visual mathematics and pattern seeking are particular strengths.

Snapshot – Identifying Multiples

Working in pairs, students color in all the multiples of 2 on Chart A and all the multiples of 4 on Chart B. They also color the multiples of 3 on another chart.

The teacher displays these two examples of student work and begins the whole-class conversation by asking, “What do you notice, what do you wonder about these two charts?”



Students respond with their observations, and these are recorded on the whiteboard:

- “There are more numbers colored in on Chart A than on Chart B.”
- “They were really careful with their coloring – it looks pretty!”
- “It makes a pattern.”
- “All the numbers we colored in are even numbers.”
- “On Chart A it goes by twos and on B it goes by fours.”
- “Chart A looks like a checkerboard.”
- “Chart B is sort of like that, too, but the coloring doesn’t go all the way across some rows.”
- “All the numbers colored on Chart B are colored in on Chart A, too.”

The goal of this segment of the lesson is for students to examine, make sense of, and offer conjectures to explain why there are half as many multiples of 4 as there are multiples of 2 (SMP.1, 3, 6, 7, 8). Based on the students’ observations, the teacher poses a series of questions and prompts for students to investigate, which include:

- How do we know if we found all the multiples on each chart? Convince us.
- Why is it that all the multiples of 2 and all the multiples of 4 are even numbers?
- Why are there more multiples of 2 than multiples of 4 on our charts?
- You noticed some patterns. Let’s think about why the multiples look like a pattern.

- Why does Chart A look like a checkerboard? What does that tell us?
- Why didn’t all the numbers in a row such as the sixes row on Chart B get colored in?

The teacher provides a structure for students to talk in small groups, addressing one or two of the questions posed. (See the snapshot *Peer Revoicing* in chapter 2 as well as the vignette ***Productive Partnerships****.* The teacher anticipated the discussion and purposefully selected questions to support student engagement. During the peer interactions, the teacher visits each of the groups to observe and listen as students collaborate. This allows the teacher informal, formative assessment opportunities that guide the discussion, support the use of academic vocabulary, and pose additional probing questions as needed.

(end snapshot)

Fourth-grade students “round multi-digit numbers to any place” (4.NBT.3). Without a deep understanding of place value, rounding a large number makes no sense, and students often resort to rounding numbers based merely on a set of steps or rules to follow. Third-grade students, asked to round 8 to the nearest 100, did not consider that this would mean rounding to zero. On a parallel task for fourth-graders from Illustrative Mathematics (Illustrative Mathematics, n.d.b), Rounding to the Nearest 100 and 1000, students with limited understanding of place value are able to round 791 to the nearest 1000 but are less successful with rounding 80 to the nearest 1000. Frequent and thoughtful use of context-based estimation can support students’ understanding of rounding (SMP.7, 8).

Estimation can often be overlooked in favor of algorithms that produce exact answers. However, estimation is a powerful, and often more practical, skill whose development can benefit students’ number sense and ingenuity in calculations. Moreover, estimation can often be carried out efficiently as a mental computation and so lends itself as a quick check that students can employ before, during, and after using precise but more cumbersome techniques. By explicitly focusing on estimating as a valuable skill in its own right, students can move beyond rounding or guessing and into strategies that make use of the structure and properties of numbers. When students have a legitimate purpose to estimate, a problem that emerges from an authentic situation, the concept of estimation has real meaning. Students might estimate how many gallons of juice to purchase for an upcoming school event, the amount of time needed to walk to the public library, the amount of wall space that can be painted with a quantity of paint, or the budget needed to create a garden on campus.

Snapshot – Estimating

Mr. Handy's class has asked the school principal, Ms. Jardin, for funding to create a vegetable garden on campus. Their proposal pointed out that the students would grow healthy vegetables that could be part of school lunches and requested enough money to buy the materials needed: fencing, boards, and nails to build planter beds, garden soil, a long hose, a few tools, and seeds. Ms. Jardin responded that she is interested in the proposal and is willing to ask the school board for funds if the student council will provide an estimate of the costs. She will need the cost estimate quickly, however, in time for the next school board meeting.

In small groups, the fourth-graders excitedly discuss ways to create a reasonable estimate of costs. They list considerations:

1. What will the dimensions of the garden be, and how much fencing is needed?
2. How many planter beds will we have and how large will they be?
3. How many tools will we need? Which tools?
4. How long will the hose need to be?
5. Which seeds will we choose and how many packages should we buy?
6. What is the price of:
    a. fencing?
    b. boards for planter beds?
    c. garden soil?
    d. tools?
    e. hose?
    f. seeds?

Mr. Handy circulates, listening as groups discuss and noting meaningful ideas on a list. In a whole-group debrief, he shares the emerging list and guides the groups to reach consensus. Aware that students sometimes believe that calculating exactly is “better” than estimating, Mr. Handy reminds students that the goal is a reasonable *estimate*, not an exact amount, and that time is limited. After a brief discussion, the class concludes that in this circumstance, approximation is preferable to calculation. Mr. Handy assigns each group member the responsibility of finding prices and estimating how much would be needed of a specific item. He further advises that, as the groups determine reasonable quantities and prices, they should round these numbers to the nearest tens or hundreds place as appropriate.

Students use online resources to search for reasonable prices for the items and work collaboratively to determine reasonable estimates. They bring their results to Mr. Handy, who reviews ideas and consults with any groups needing additional support. Once estimates are ready for submission, each group records their recommendations on a shared spreadsheet. The students conclude the lesson with great enthusiasm and anticipation of a successful outcome for their proposal.

(end snapshot)

Real-world problems rooted in local context matter when supporting students’ understanding of mathematics content. Memorizing rules about whether to round up or down based on the last digits of a number may produce correct responses some of the time, but little conceptual development is accomplished with such rules.

Grade Five

Fifth grade marks the last grade level at which Number and Operations in Base Ten is an identified domain in the CA CCSSM. At this grade, students work with powers of 10, use exponential notation, and can “explain patterns in the placement of the decimal point when a decimal is multiplied by a power of 10” (5.NBT.2). Fifth-grade students are expected to fully understand the place-value system, including decimal values to thousandths (SMP.7; 5.NBT.3). The foundation laid at earlier grades is of paramount importance in a fifth-grader’s accomplishment of these standards.

To build conceptual understanding of decimals, students benefit from concrete and representational materials and consistent use of precise language (Carbonneau, Marley, and Selig, 2013). When naming a number such as 2.4, it is imperative to read it as “2 and 4 tenths” rather than “2 point 4” in order to develop understanding and flexibility with number. Base-10 blocks are typically used in the primary grades, with the small cube representing one whole unit, a rod representing 10 units, and a 10 x 10 flat representing 100. If instead, the large, three-dimensional cube is used to represent the whole, students have a tactile, visual model to consider the value of the small cube, the rod, and the 10 x 10 flat. Another useful tool is a printed 10 x 10 grid. Students visualize the whole grid as representing the whole and can shade in various decimal values. For example, if two columns plus an additional five small squares are shaded on the grid, the student can visualize that value as 1.25 or 1-1/4 of the whole. When decimal numbers are read correctly—for example, reading .25, as “twenty-five hundredths”—students can make a natural connection between the decimal form and the fractional form, noting that “twenty-five hundredths” can be written as the fraction 25/100, which simplifies to 1/4 (SMP.6).

Fifth-grade students use equivalent fractions to solve problems; thus, it is essential that they have a strong grasp of equality (SMP.6) and have developed facility with using benchmark fractions (e.g., 1/2, 2/3, 3/4) to reason about, compare, and calculate with fractions. Experiences with placing whole numbers, fractions, and decimals on the number line contribute to building fraction number sense. Students need time and opportunity to collaborate, critique, and reason about where to place the numbers on the number line (SMP.2, 3). For example, where might 4/7 be placed in relation to 1/2? As students advance to middle school mathematics, their understanding of place value and flexibility with whole numbers, fractions, and decimals will prepare them to work successfully with integers, percents, and ratios.

How Do Children in Grades Three Through Five Develop Understanding of the Operations of Multiplication and Division?

Grade Three

Building understanding of multiplication and division constitutes a large part of the content for third grade. These students first approach multiplication as repeated addition of equal size groups, such as the illustrations here, which show 4 groups of 3 stars, for a total of 12 stars: 4 × 3 = 12.

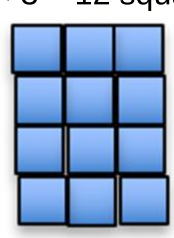
Repeated Addition: 4 × 3 = 12

Then, as they apply multiplication to measurement concepts, students begin to view multiplication as “jumps” on a number line, as well as in terms of **arrays** and area.

4 × 3 = 12 on a number line

Array, 4 × 3 = 12

Area, 4 × 3 = 12 square units



Students who make sense of numbers are likely to develop accurate, flexible, and efficient methods for multiplication. For example, to multiply 8 × 7, a student may find an easy approach by decomposing the 7 into 5 + 2 and thinking 8 × 5 = 40; 8 × 2 = 16; 40 + 16 = 56. Children with well-developed number sense readily make successful use of the distributive property (SMP.7; 3.OA.5).

Grade Four

Concepts of multiplication advance in fourth grade, when students first encounter multiplication as comparison. Problems now include language such as "three times as much" or "twice as long." Students need to be able to make sense of such problems and be able to illustrate them (SMP.1, 5). Strip diagrams, number lines, and drawings that represent a story's context can support students as they develop understanding. This knowledge will serve them well as they begin to solve fraction multiplication problems, in which comparison contexts are frequently involved.

To multiply multi-digit numbers with understanding (4.NBT.5), fourth-graders need to have internalized place-value concepts. When thinking about 4 × 235, for example, students can use front-end estimation to recognize that the product will be greater than 800, because 4 × 200 = 800. Students who consistently and intentionally use mathematical practices (SMP.1, 2, 6) will continue to make sense of multiplication as larger quantities and different contexts and applications are introduced. See the vignette *Grade Four: Multiplication*, which illustrates a lesson where the teacher strengthens student understanding of multiplication as comparison.

Grade Five

Understanding place value and how the operations of multiplication and division are related allows fifth-grade students to "find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors" (5.NBT.6). A student can solve 354 ÷ 6 by decomposing 354 and dividing each part by 6, applying the distributive property. Thinking that 354 = 300 + 54, they can divide 300 by 6, and then 54 by 6 mentally or with paper and pencil: 300 ÷ 6 = 50; 54 ÷ 6 = 9, and 50 + 9 = 59. Therefore, 364 ÷ 6 = 59. Or a student could use multiplication to solve 354 ÷ 6 by thinking 60 × 6 =

360, and then considering that 59 × 6 = 360 – 6, and 360 – 6 = 354. In words, the student can express that it takes 60 sixes to make 360, and it would take one less 6 (59 rather than 60) to make 354. Ample experience with math talks exposes students to a rich variety of mental strategies and positions them to select wisely from their repertoire of methods to apply a particular strategy in a given problem situation. It is essential that students have developed a robust understanding of the operations of multiplication and division as they approach the middle grades, where they will apply such reasoning to solve ratio and rate problems.

How Do Children in Grades Three Through Five Come to Make Sense of Operations with Fractions and Decimals?

The grade five standards state that students will "Apply and extend previous understandings of multiplication and division to multiply and divide fractions" (5.NF.3 – 7). This is a challenging expectation and deserves attention at every grade level. The story problems and tasks children experience in the younger grades typically rely on contexts in which things are counted rather than measured to determine quantities ("how *many* apples, books, children…" rather than "how *far* did they travel, how *much* does it weigh…"). However, measurement contexts more readily allow for fractional values and support working with fractions. A student who solves a measurement problem involving whole numbers can apply the same reasoning to a problem involving fractions. For example, weights of animals can serve as the context for subtraction comparisons (e.g., *Our dog weighs 28 pounds and our neighbor's dog weighs 34 pounds. How much more does the neighbor's dog weigh than our dog?*), and the same thinking is needed if weights involve decimals or fractions (28.75 pounds vs. 34.4 pounds). The use of decimals and fractions makes it possible to describe situations with more precision.

To support students making connections between operations with whole numbers and operations with fractions, teachers should emphasize a greater balance between "counting" and "measuring" problem contexts throughout transitional kindergarten through grade five. (See chapter 6 for additional discussion and examples of fraction concept development.)

Grade Three

A major component of third-grade content is the introduction of fractions as a number. Previous grade-level work includes exploring fractions in geometric shapes and time. Students focus on understanding fractions as equal parts of a whole and as numbers located on the number line, and they use reasoning to compare unit fractions (3.NF.1, 2, 3). Particular attention needs to be given to developing a firm understanding of 1/2 as a basis for comparisons, equivalence, and benchmark reasoning. Students might explore the idea of the whole and equal parts with Cuisenaire rods. In tasks such as Locating Fractions Less than One on the Number Line, found at Illustrative Mathematics (Illustrative Mathematics, n.d.c), students partition the whole on a number line into equal halves, fourths, and thirds and locate fractions in their relative positions.

Grade Four

At this grade, students develop an understanding of fraction equivalence by illustrating and explaining their reasoning. Students can strengthen their knowledge of fraction equivalence by engaging in games that provide practice, such as Matching Fractions or Fractional Wall, created by Nrich Maths (University of Cambridge, n.d.). Fourth-graders add and subtract fractions with like denominators, relying on the understanding that every fraction can be expressed as the sum of unit fractions. 7/4, then, can be expressed as 1/4 + 1/4 + 1/4 + 1/4 + 1/4 + 1/4 + 1/4. The Number and Operations–Fractions 3–5 Progression reiterates the importance of students building their understanding of unit fractions. “Initially, diagrams used in work with fractions show them as composed of unit fractions, emphasizing the idea that a fraction is composed of units just as a whole number is composed of ones” (Common Core Standards Writing Team, 2022, 135).

Students in these grades come to recognize that a unit fraction is a *number*; it is something they can count in the ways they count and add with whole numbers. They can determine, for example, that two one-fourths plus three one-fourths equal five one-fourths, or 5/4. Further, by using unit fractions to build other fractions, students begin to make sense of adding and subtracting fractions with unlike denominators. This understanding allows them to “apply and extend previous understandings of multiplication to multiply a fraction by a whole number (4.NF.4)” when solving word problems. They represent their thinking with diagrams (e.g., number lines and strip diagrams), pictures, and equations (SMP.2, 5, 7). This work lays the foundation for further operations fractions in fifth grade.

Grade Five

Fifth-grade students apply their understanding of equivalent fractions to add and subtract fractions with unlike denominators (5.NF.1). They multiplied fractions by whole numbers in fourth grade; now they extend their understanding of multiplication concepts to include multiplying fractions in general (5.NF.4). Division of a whole number by a unit fraction (12 ÷ 1/2) and division of a unit fraction by a whole number (1/2 ÷ 12) are challenging concepts that are introduced in fifth grade (5.NF.7). To make sense of division with fractions, students must rely on an earlier understanding of division in both partitive (fair share) and quotitive (measurement) situations for whole numbers. The terms “partitive” and “quotitive” are important for teachers’ understanding; students may use the less formal language of fair share and measurement. What is essential is that students recognize these two different ways of thinking about division as they encounter contextual situations. Fifth-grade students who understand that 12 ÷ 4 can be asking “how many fours in 12?” (a quotitive view of division) can use that same understanding to interpret 12 ÷ 1/2 as asking “how many 1/2s in 12?” (Van de Walle et al., 2014, 235). Applying understanding of operations with whole numbers to the same operations with fractions relies on students’ use of sophisticated mathematical reasoning and facility with various ways of representing their thinking (SMP.1, 5, 6).

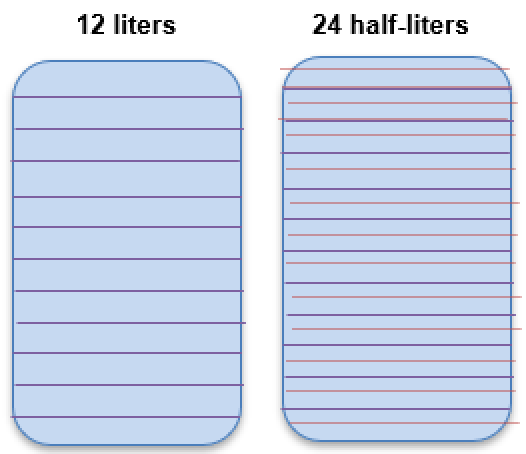
How might fifth-grade students approach a problem such as this?

> *To make banners for the celebration, the teacher bought a 12-yard roll of ribbon. If each banner takes 1/2 yard of ribbon, how many banners can be made from the 12-yard roll of ribbon?*

A quotitive interpretation of division and a number line illustration can be used to solve this problem. If a length of 12 yards is shown, and 1/2-yard lengths are indicated along the whole 12 yards, the solution, that 24 banners can be made because there are 24 lengths of 1/2 yard, becomes visible.

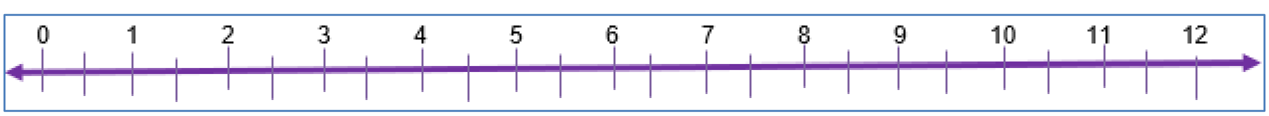
Now consider the following problem:

> *For the foot race in the park tomorrow, our running coach bought a 12-liter container of water. We plan to fill water bottles for the runners. We will pour 1/2 liter of water into each bottle. How many bottles can we fill? Will we have enough water for all 28 runners?*



A quotitive interpretation of division and a picture or a number line illustration can be used to solve this problem. The student begins by illustrating a quantity of 12 liters. The student then marks 1/2-liter sections horizontally and finds there are 24 half liters.

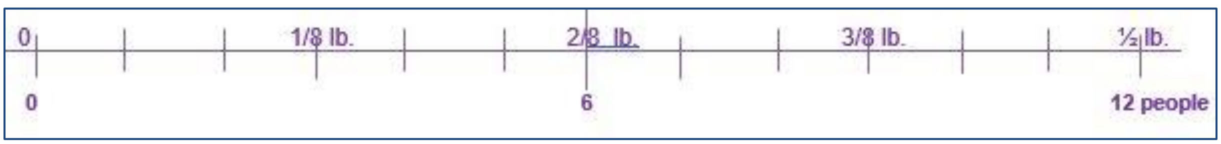
A number line illustration:



In either case, students can visually recognize that 24 water bottles can be filled because there are 24 half-liters in 12 whole liters (SMP.1, 2, 4, 5, 6).

To understand what 1/2 ÷ 12 means as **partitive division**, a suitable context might involve 1/2 pound of candy to be shared among 12 people and asking how much each person would get. A picture or number line representation can be used to illustrate the story. The solution can be seen by separating the 1/2 pound into 12 equal parts and finding that each portion represents 1/24 of a pound of candy.

Sense-making for fraction division becomes accessible when students discuss their reasoning about problems set in realistic contexts and use visual models and representations such as the following to express their ideas to others (SMP.1, 3, 6).



Third- through fifth-grade students who can make sense of operations with fractions and decimals, who can analyze a contextual situation involving fractions, and who can represent their thinking are prepared for the middle school expectation that they:

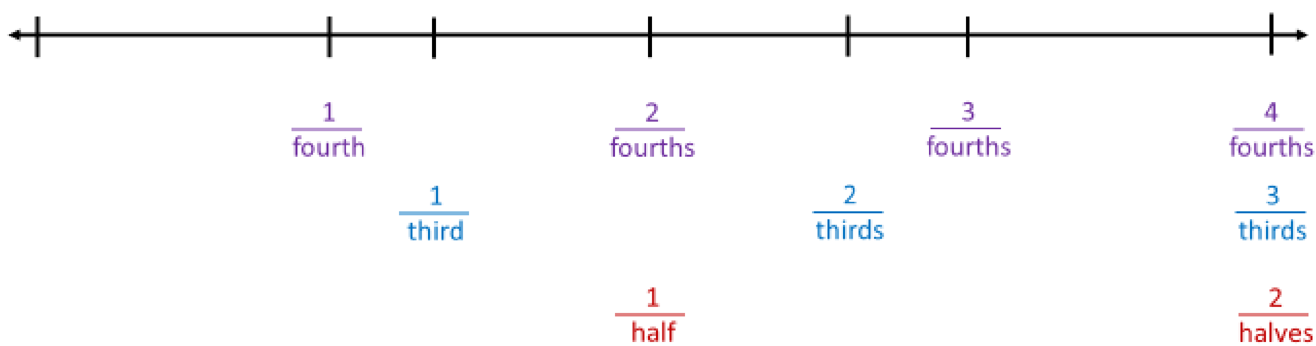
- apply and extend previous understandings of multiplication and division to divide fractions by fractions (6.NS),
- fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation (6.NS.3), and
- apply and extend previous understandings of arithmetic to algebraic expressions (6.EE).

How Do Students in Grades Three Through Five Use Number Lines as Tools?

Grade Three

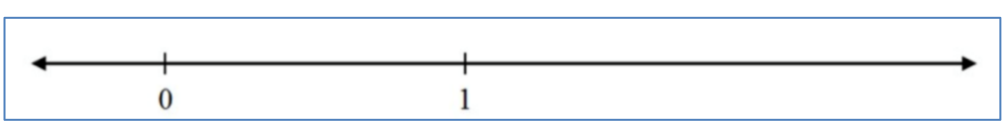
Younger-grade students use number lines to order and compare whole numbers and to illustrate addition and subtraction situations. In third grade, children extend their reasoning about numbers. They begin using number lines to represent fractions and to solve problems involving measurement of time (3.NF.2; 3.MD.1; SMP.3, 5). In the first and second grades, students partitioned shapes into equal parts and described these parts with words: halves, thirds, fourths, etc., but they did not write fractions as numbers, 1/2, 1/3, 1/4, etc. (1.G.A.3; 2.G.A.3).

Third-graders begin to record fractions as numbers and to locate fractions on the number line (3.NF.A.1, 2; SMP.2, 6, 7). The concepts of numerator and denominator are new to students and crucial to understanding of fractions. Writing the denominators of fractions in word form (as in the illustration below) initially can help students distinguish between numerators and denominators and serves to link their previous understanding of fractional parts with the more abstract idea of fractions as numbers on a number line. The denominator of a fraction tells the name of the piece, and this understanding enables students to make sense of why, when adding fractions, it is necessary for the fractions to have the same denominator.



Third-grade students use reasoning about the relative sizes of fractions to estimate their positions on the number line. For example, in the third-grade task Find 1/4, Starting From 1 from Illustrative Mathematics (Illustrative Mathematics, n.d.d), students need to determine where 1/4 is located. This calls for understanding that 1/4 means one of four equal parts, and that we can represent that quantity as a location on the number line, one-fourth the distance between 0 and 1 whole.

The number line shows two numbers, 0 and 1:



Where is 1/4 on this number line?

Grade Four

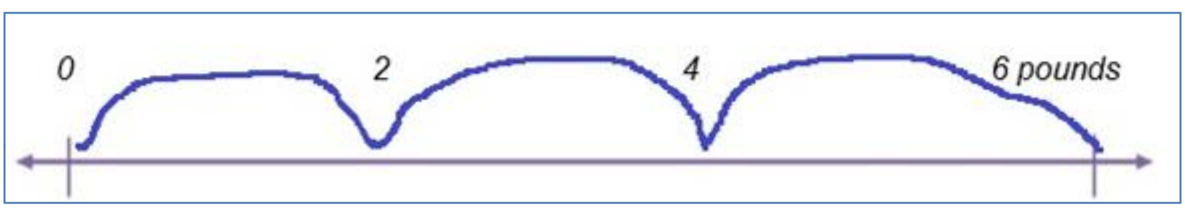
Fourth-graders develop facility with naming and representing equivalent fractions and begin to use decimal notation for fractions. They continue to build their capacity to locate and interpret values on a number line (4.NF.1, 2, 6, 7; SMP.1, 5, 7). Students can find equivalent names for fractions, determine the relative size of fractions and decimal fractions, and use reasoning to locate these numbers on a number line. For example, a task might provide a number line on which the numbers 2.0 and 2.5 are identified, and students use their understanding of fractions to locate 1.0, 0.75, 5/4, 7/3, and 1 8/10.

Grade Five

Fifth-graders apply strategies and understandings from previous grade-level experiences with multiplication and division to make sense of multiplication and division of fractions (5.NF.6, 7c; SMP.1, 2, 5, 6). This includes using the number line as a tool to represent problem situations. Multiplication and division with fractions can be conceptually challenging. By making explicit connections between thinking strategies and representations previously used for whole number multiplication and division, teachers can support students' developing understanding of these operations.

Whole number example:

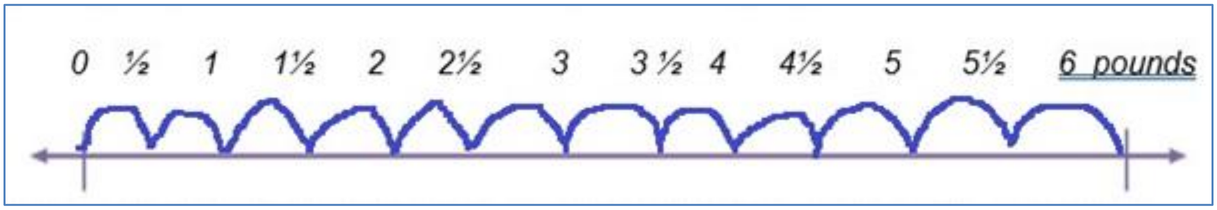
We harvested 6 pounds of radishes in our garden and put 2 pounds into each basket. How many baskets did we use?



We used three baskets. (Note the 2-pound jumps above, starting at 6 and working backwards along the number line to represent the three baskets needed.)

Parallel fraction example:

We harvested 6 pounds of radishes in our garden. We put radishes into bags, placing 1/2 pound of radishes in each bag. How many bags did we fill?



Using the same strategy as before, we can see that we filled 12 bags. (Note the equal 1/2-pound jumps, starting at 6 and working backwards along the number line to represent 12 bags of radishes.)

Extensive and thoughtful experience with locating whole numbers and fractions on the number line in grades three through five will position students for success in grades six through eight mathematics work with the system of rational numbers. In middle grades, students will place positive and negative values on the number line, apply previous understandings of addition and subtraction to rational numbers, and graph locations in all four quadrants of the coordinate plane (6.NS.6, 7, 8; 7.NS.1).

Middle Grades, Six Through Eight

As students enter the middle grades, the number sense they acquired in the elementary grades deepens with the content. Students transition from exploring numbers and arithmetic operations in kindergarten through grade five to exploring relationships between numbers (CC2, Exploring Changing Quantities; and CC3, Taking Wholes Apart and Putting Parts Together) and making sense of contextual situations using various representations. SMP.2 is especially critical at this stage, as students represent a wide variety of real-world situations through the use of real numbers and variables in expressions, equations, and inequalities.

Three big ideas (Boaler, Munson, and Williams, 2018) related to number sense for these grades call for students to do the following:

- Demonstrate number line understanding
- Develop an understanding of ratios, percents, and propositional relationships

- See generalized numbers as leading to algebra

How Is Number Line Understanding Demonstrated in Grades Six Through Eight?

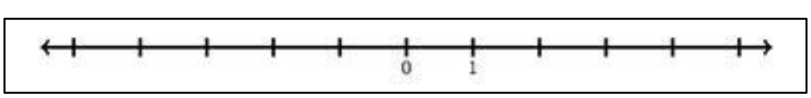
Grade Six

Number lines are an essential tool for teachers to help students create a visual understanding for numbers. Work with number lines begins in second grade as students use them to count by positive integers, to determine whole number sums and differences, and as a distance model and measurement tool with a ruler. By third grade, students use number lines to place and compare fractions as well as solve word problems. In fourth grade, the use of number lines includes decimals. In fifth grade, students use number lines as a visual model to operate with fractions. They are also introduced to coordinate planes in fifth grade. In sixth grade, rational numbers, as a set of numbers that includes whole numbers, fractions, decimals, and their opposites, are seen as points on a number line (6.NS.6) and as points in a coordinate plane (6.NS.6.b and c), which expands on the fifth-grade view of coordinate planes. Ordered pairs, in the form a,b, are introduced as the notation to describe the location of a point in a coordinate plane. Sets of numbers can often be efficiently represented on number lines, and, at the sixth-grade level, students are introduced to the strategy of representing solution sets of inequalities on a number line (6.EE.8).

Students also see the relationship between absolute value of a rational number and its distance from zero (6.NS.7.c) and use number lines to make sense of negative numbers, including in contexts such as debt. The task below demonstrates an example of how number lines can be used to achieve an understanding of the connection between opposites and positive/negative.

Task (adapted from Illustrative Math, “Integers on the Number Line 2”)

Below is a number line with 0 and 1 labeled:



We can find the opposite of 1, labeled -1, by moving one unit past 0 in the opposite direction of 1. In other words, since 1 is one unit to the right of 0 then -1 is one unit to the left of 0.

1. Find and label the numbers -2 and -4 on the number line. Explain.
2. Find and label the numbers $-(-2)$ and $-(-4)$ on the number line. Explain.

As two quantities vary proportionally, double number lines capture this variance in a dynamic way. Grade six students are introduced to the strategy of using double number lines to represent whole number quantities that vary proportionally (6.RP.3). The vignette *Mixing Paint* in chapter seven provides an illustration of the double number line strategy for a sixth-grade ratio and proportion problem.

Grade Seven

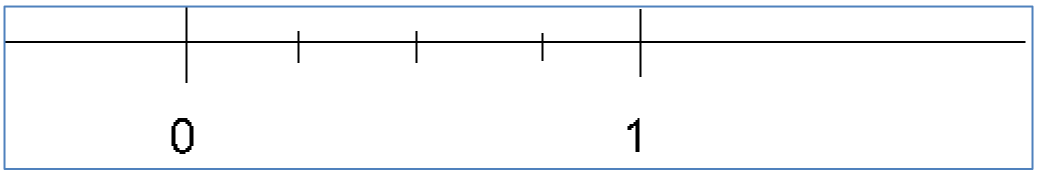
In seventh grade, students develop a unified number understanding that includes all types of numbers they have seen in previous standards. That is, they understand fractions, finite or repeating decimals, percents, and integers as different representations of rational numbers, and they attend to precision in their use of these words (SMP.6). Every fraction, finite or repeating decimal, percent, and integer can be written in the form of a/b where a is an integer and b is a positive integer—and understandings of fractions, decimals, percents, integers, and whole numbers can all be subsumed into a larger understanding of rational numbers. This unified understanding is achieved, in part, through students' use of number lines to represent operations on rational numbers, such as the addition and subtraction of rational numbers on a number line (7.NS.1).

For students, the mechanics of using a number line to represent operations on rational numbers rests upon two realizations: first, rational numbers are locations on the number line; and second, the distances between rational numbers are also rational numbers. Teachers should use activities that promote the understanding of these two realizations. For the addition of two rational numbers, for example, the first number can be seen as fixing a location, while the second number refers to the distance moved away from the first number. The following snapshot illustrates this relationship.

Snapshot: Visualizing Fractions on and Within a Number Line

Ms. V knows that her students struggle with labeling fractions on a number line. She poses the following task to them:

> *In looking at the number line diagram below, the quantity 1/4 appears more than once. Talk with your partner about all the ways 1/4 occurs in the diagram. How many can you and your partner come up with?*



Most student pairs recognize that the first tickmark to the right of 0 can be labeled with 1/4. The pairs struggle in coming up with a second place that 1/4 is seen. Ms. V asks them if they can label the other tick marks. They can see that the middle tickmark can be labeled as 1/2. Ms. V then encourages them to think of 1/2 as 2/4. One pair excitedly raises their hand “there is another 1/4 to get from 1/4 to the 2/4!” Ms. V asks them where this appears on the diagram, and one of the pair places it between the 1/4 and 2/4 tickmarks. The other students offer the other “between tickmark” places as other appearances of 1/4. Thus, they see that 1/4 only occurs once, as a location, but it occurs four times as a distance or length.

This two-fold usage of number lines to represent locations and distances is used to solidify further ideas: opposite quantities, known as additive inverses, combine to make 0 (7.NS.1a); subtraction is actually addition of an additive inverse, and the distance between two rational numbers is the absolute value of their difference (7.NS.1c). In bringing attention to numbers as serving as both locations and distances, Ms. V has given her students more tools to help them explore how quantities, and the changes between them (CC2), can be represented on a number line.

(end snapshot)

Seventh-graders also extend the use of double number lines that represent whole number quantities (introduced in sixth grade, 6.RP.3) to now include fractional quantities that vary proportionally (7.RP.1). The vignette Grade Seven, Using a Double Number Line, illustrates how a teacher supports students in building this extension.

Grade Eight

In eighth grade, students' understanding of rational numbers is extended in two important ways. First, rationals have decimal expansions that eventually repeat, and, vice versa, all numbers with decimal expansions that eventually repeat are rational (8.NS.1). A typical task to demonstrate the first aspect of this standard is to ask students to use long division to demonstrate that 3/11 has a repeating decimal expansion, and to explain why. As students realize the connection between the remainder and the repeating portion (once a remainder appears a second time, the repeating decimal is confirmed), their understanding of rational numbers can now more fully integrate with their understanding of decimals and place value.

Second, as students begin to recognize that there are numbers that are not rational—*irrational* numbers—they can see that these new types of numbers can still be located on the number line and that these new irrational numbers can also be approximated by rational numbers (8.NS.2). The foundation for this recognition is actually built through seventh-grade geometry explorations of the relationship between the circumference and diameter of a circle and formalized into the formula for circumference (7.G.4), where the division of the circumference by the diameter for a given circle always results in a number a little larger than 3, irrespective of the size of circle. Of course, in exploring this quotient of circumference by diameter, students get a look at a decimal approximation for their first irrational number, pi. This groundwork in quotients is critical, as students use rational approximations (an integer divided by an integer) to compare sizes of irrational numbers, locate them on number lines, and estimate values of irrational expressions, like pi^2.

The think-pair-share format can be used as a powerful means to build number sense for this new type of number, irrational numbers, as illustrated in the following snapshot.

Snapshot: Grade Eight, Irrationals on a Number Line

Ms. H designs a lesson for her students to see that irrational numbers behave much like rational numbers, in that they can be taken apart and “repackaged” in ways that, though more symbolic, rely upon the same properties as rational numbers (CC3). She has decided to build on a short think-pair-share activity for her students to engage with classmates to place rational and irrational numbers on a number line (8.NS.2). Ms. H begins: “Please copy this number line on the board onto your paper. I would like for you to spend a minute or so thinking quietly about where to place sqrt(4) and sqrt(9) on your number line. When your thinking is complete, talk with a partner about why you decided on your number line placements.”

Ms. H walks between students monitoring work, asking questions to promote the use of academic vocabulary and align her instruction with ELD support for English learners. She encourages all of her students to use open sentence frames (“I placed sqrt(4) here because [blank]” or “Since sqrt(9) equals [blank], then I placed it [blank]”) to expand their use of mathematical language. She supports her linguistically and culturally diverse English learners, observing and listening to them speak about where to place the values while paying close attention to their use of mathematical language and providing additional guiding questions, judicious coaching, and corrective feedback when necessary. In providing designated ELD support, she provides lists of terms related to the language of comparison, such as “the same as,” “close to,” “almost,” “greater than,” “less than,” “smaller,” and “larger.” (See chapter two for more on UDL and ELD strategies.)

Ms. H: “Oh, I see many of you recognized that these values are more simply expressed as our good friends 2 and 3! Next, I want to give you another minute for you to place sqrt(5) on the number line.”

(After 60 seconds or so)

Ms. H: “Okay, please check with your partner. How do your locations compare?”

(Conversation in pairs)

Ms. H: “Can someone describe how they placed sqrt(5) on their number line using the document camera?”

(Several pairs show their placement and describe their thinking.)

Ms. H: “Lastly, please describe how to determine where 2*sqrt(5) should be placed. Think about this on your own for a minute or so, then check with your partner.”

(Students work individually then in pairs on this extension of their previous work, finally sharing their work when finished.)

(*end snapshot)*

Irrational numbers other than pi, such as sqrt(2), can be introduced in eighth grade in a concrete geometric way, such as the following activity to be done on a pegboard with rubber bands:

1. Using a rubber band, create a square with area 4.
2. Now draw a square with area 9.
3. Can you draw a square with area 2?
4. How about drawing a square with area 5? Area 3?

How Do Students in Grades Six Through Eight Develop an Understanding of Ratios, Rates, Percents, and Proportional Relationships?

Grade Six

In sixth grade, students are introduced to the concepts of ratios and unit rates (6.RP.1 and 6.RP.2) and use tables of equivalent ratios, double number lines, tape diagrams, and equations to solve real-world problems (6.RP.3). A critical feature to emphasize for students is the ability to think multiplicatively rather than additively. For example, in the table below, missing values in a column can be found by multiplying (or dividing) a different column by a number; in the same table, moving from the second column (with 10 cups of sugar) to the third column (with 1 cup of sugar) requires dividing by 10, so this same calculation is done in moving from 16 cups of flour to 1.6 cups of flour. Alternatively, in moving between rows, students can see that multiplying (or dividing) by a number is used in moving from the cups of sugar to cups of flour; in the case below, multiplying the cups of sugar by 1.6 results in the appropriate cups of flour in the second row.

Cups of sugar	5	10	1		1.5	15	
Cups of flour	8	16		0.8	2.4		

Presenting scenarios where students must recognize whether two quantities are varying additively (same amount added/subtracted to both) or multiplicatively (both quantities are multiplied/divided by the same value) can strengthen proportional reasoning, which follows in later grades. As students work with covarying quantities, such as miles to gallons, they see the value in expressing this relationship in terms of a single number that represents a unit rate, miles per (one single) gallon or miles per gallon.

Grade Seven

In seventh grade, students' understanding of rates and ratios is drawn upon to recognize and represent proportional relationships between quantities (7.RP.2). There are a host of representations for students to be introduced to, and to later draw from, as they reason through proportional situations: graphs, equations, verbal descriptions, tables, charts, and double number lines. Although there are many approaches to solving proportions, the emphasis in any approach should always be on sense making rather than on "answer getting," as the box below further explains.

Pitfalls with Proportions

There is a danger in working with proportions for students to shift away from sense-making to "answer-getting," as Phil Daro points out (Daro, 2014). One classic case of this is in the use of cross-multiplication to solve for unknowns in a proportion. For example, an elementary school wishes to determine the number of swings needed at recess on the playground. Not all students swing, so it is determined that, at a minimum, two swings are needed for every 25 students. At recess, how many swings, at a minimum, are needed for 150 students? A typical approach to this would be to set up a proportion as

$$(2\ \text{"swings"})/(25\ \text{"students"})=(x\ \text{"swings"})/(150\ \text{"students"})$$

In solving for the number of swings, students are often led to cross-multiply then divide to find the unknown:

$$2\cdot 150=25\cdot x$$

$$300=25\cdot x$$

$$12=x$$

Although this leads to a correct answer, there are several pitfalls associated with cross-multiplying: The units become nonsensical when multiplied (the units label for 300 in the second equation is…swing-students?).

Once introduced to cross-multiplying, students are strongly visual, so whenever they see two fractions, regardless of the operation or relationship between them, they are inclined to cross-multiply as a way to “eliminate” the fractions at the outset. Thus, cross-multiplying can contaminate, or even circumvent, sensible strategies to perform operations with fractions.

As pointed out earlier, sense-making should be an emphasis, and algorithms should be used after students have developed conceptual understanding (Lamon, 2012; Siegler et al., 2010). Cross-multiplying eschews approaches such as scaling up or recognizing internal factors, which contribute to greater number sense and provide means for students to explore changing quantities meaningfully (CC2).

Initially, students test for proportionality by examining equivalent ratios in a table or by graphing the relationship and looking for a line (7.RP.2.a). They may also attempt to identify a constant of proportionality (7.RP.2.b) or represent the equation as a relationship (7.RP.2.c). Although percents are introduced in sixth grade, percents are often used in the context of proportional reasoning problems in seventh grade (7.RP.3). Because of the rich variety in approaches to solving proportional problems, teachers should make good use of class conversations about open-approach problems. The vignette *Grade Seven, Ratios and Orange Juice* provides an example of an open-approach problem involving ratios.

Grade Eight

Understanding of proportional relationships plays a fundamental role in helping students make sense of linear equations graphically. In plotting points and drawing a line, students recognize that each graph of a proportional relationship between two quantities is actually a line through the origin, and that the unit rate, in units of the vertically oriented quantity (y) per one unit of the horizontal quantity (x), is the slope of the graph (8.EE.5). By situating the graphical features of a line, such as the slope, in prior understanding of proportions, students are able to internalize an understanding of linear equations that is interwoven with their understanding of contexts for linear equations, as opposed to two disconnected schemas. The following task can provide a means to connect ratio tables, unit rates, and linear relationships.

Task – Unit Rates, Line and Slope

Two cups of yellow paint are mixed with three cups of blue paint to make Gremlin Green paint.

a. How much yellow and blue paint is needed to make 35 cups of the Gremlin Green paint?

b. Set up a ratio table that shows all three pairs of unit rates.

c. Write two unit rate statements based on your work in part a.

d. Choose two points from your ratio table and graph the line through these points. How does the slope of your line relate to the unit rates in your table from part b?

How Do Students in Grades Six Through Eight See Generalized Numbers as Leading to Algebra?

Grade Six

To many, algebra is seen as a type of generalized arithmetic, with letters as stand-ins for general numbers in expressions (Usiskin, 1999). In sixth grade, students are introduced to the idea that letters can stand for numbers (i.e., using a letter for a nonspecific, general number); they write, read, and evaluate expressions involving letters, operations, and numbers (6.EE.1). For sixth-grade students, variables are intrinsically related to numbers, and the conceptions they have formed about how numbers operate form the basis of their understanding of how variables operate. As students take apart expressions and put parts together in building different expressions, first with numbers, then with variables, they further their understanding of the fundamental idea of Taking Wholes Apart and Putting Parts Together (CC3).

Ideas of equivalence and operations, laid before in earlier grades, now take on new meaning as students apply properties of operations to generate equivalent expressions (6.EE.3) and identify when two expressions are equivalent (6.EE.4). Additionally, the relationship between numerical understanding and algebraic understanding is reciprocal; for example, the recognition that $t + t + t$ is equivalent to $3t$ can provide additional insight for students to see multiplication as repeated addition. The number sense children have developed to this point also enables them to go beyond building and comparing expressions to reasoning about and solving one-variable equations of various types (6.EE.7).

Grade Seven

Students' understanding of rational numbers, as whole numbers, fractions, decimals, and percents, supports their ability to solve real-life and mathematical problems in seventh grade (7.EE.3). Specifically, students construct (from word problems) and solve equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are rational numbers in seventh grade (7.EE.4). Many of the properties that students use in solving these types of equations are reliant upon a well-developed number sense. In other words, to solve equations involving unknowns that are rational numbers, students must rely upon their understanding of rational numbers themselves, at times. In the equation above, for example, students can be sure that p times x is another rational number because they have built an intuition about the closure property of multiplication through their prior work in multiplying specific rational numbers together and seeing the answers that are arrived at.

As students grow increasingly reliant upon properties—first explored with numbers in earlier grades and now seen to be consistent when letters replace numbers, such as multiplying by 1 or adding 0, to facilitate the many correct ways equations can be used to model a situation (7.EE.4.a)—their number sense develops into a sense for algebra. Because of this progression, the beginnings of algebra understanding for students should be rooted in sense-making about how numbers work in a more general setting. It is worth pointing out here that although it is tempting to provide lists of steps (e.g., simplify both sides of the equation, do the same operation to both sides, isolate the variable using operations, etc.), lists of steps should only be provided when generated by students themselves in describing their steps on particular problems, lest students trade active reasoning from intrinsic properties to a reliance upon rote procedural skills (Reys and Reys, 1998).

Grade Eight

In eighth grade, the notation for numbers expands greatly, with the introduction of integer exponents and radicals to represent solutions of equations (8.EE.2). For students with a firm grasp of numbers and variables, the introduction of this notation can be taken in stride. For example, if students are asked to compare 2 + 2 + 2 to x + x + x and to sqrt(2) + sqrt(2) + sqrt(2), the connection between these—as three twos, three xs, and three square roots of two—becomes more apparent to students and enables them to draw upon number sense in forming their algebra sense. In looking for and making use of the structure of these expressions (SMP.7), students are reacquainted with the importance of CC3 as well. Number sense also forms a critical role in eighth grade, as students can check the accuracy of their answers with estimation and use place-value understanding to express large and small numbers in scientific notation (8.EE.4).

Math Talks, Grades Six Through Twelve

Math talks, which include number talks, number strings, and number strategies, are short discussions in which students solve a math problem mentally or with a math drawing, share their strategies aloud, and as a class determine a correct solution. Number math talks can be viewed as “open” versions of computation problems because students are encouraged to invent or apply strategies that will allow them to find a solution mentally or with math drawings and to explain their approach to peers. Math talks designed to highlight a particular type of problem or useful strategy serve to advance the development of efficient, generalizable strategies for the class. These class discussions provide an interesting challenge and a safe situation in which to explore, compare, and develop strategies.

Math talks in grades six through eight can strengthen, support, and extend calculation strategies involving expressions, decimal, percent, and fraction concepts, as well as estimation. Math talks in grades nine through twelve can strengthen, support, and extend algebraic simplification strategies involving expressions, connect algebra concepts to geometry, and provide opportunities to practice estimation of answers. Also, many math talks from grades six through eight are still readily applicable in grades nine through twelve, as they can lay valuable groundwork for algebra understanding. For example, strategies that make use of place value and expanded form on multiplication problems, such as 134×36, can be employed to understand multiplication of binomials. With many of these topics, it is helpful for students to be able to use math drawings and written notations/methods to support problem solving and explanations.

The notion of using language to convey mathematical understanding aligns with the key components of the CA ELD Standards. The focus of a math talk is on comparing and examining various methods so that students can refine their own approaches, possibly noting and analyzing any error they may have made. In the course of a math talk, students often adopt methods another student has presented that make sense to them.

The CA ELD Standards promote Interacting in Meaningful Ways (26–7), where instruction is collaborative, interpretive, and productive. To facilitate meaningful discourse, the teacher can use a Collect and Display routine (SCALE, 2017). As students discuss their ideas with their partners, the teacher listens for and records, in writing, the language students use, and may sketch diagrams or pictures to capture students’ own language and ideas. These notes are displayed during an ensuing class conversation, when students collaborate to make and strengthen their shared understanding. Students are able to refer to, build on, or make connections with this display during future discussion or writing.

Some examples of problem types for math talks at the sixth- through eighth-grade level include:

- Order of operation calculations for which students can apply properties to help simplify complicated numerical expressions. For example, $3(7 - 2)^2 + 8 \div 4 - 65$.
- Operations involving irrational numbers: 2/3 of pi is approximately how much? Four times sqrt(8) is closest to which integer?
- Percent and decimal problems: Compute 45 percent of 80; or calculate the percent increase from 80 to 100; or 0.2 percent of 1000 is how much?

Some examples of problem types for math talks at the ninth- through twelfth-grade level include:

- Which graph doesn’t belong? Various collections of graphs could be used, where all but one graph agree on various characteristics. The ensuing conversations help students attend to precision in the graphs and with their language (SMP.6) as they talk out the underlying causes of the differences between the graphs. For example, four graphs of polynomial functions could be displayed, with three odd-degree polynomial and one even-degree polynomial, which can highlight the notion of how the terms *even* and *odd* are used with regards to polynomials. Another example could be where one function displayed has multiple real roots, while the others have single or no real roots.

- Rewriting expressions using radical notation, such as $(a^2b^3)^{\frac{3}{2}}$. There are often multiple approaches to simplifying expressions, so these can serve as excellent discussion points for students to see a variety of ways to approach simplification.
- Similarly, there is merit to sharing and discussing the myriad ways to approach multiplying monomials, binomials, and trinomials (e.g., (x + y)(3x – 2y)), including algebraic properties, such as the distributive property, and generic rectangles.

Games, Grades Six Through Twelve

Games are a powerful means of engaging students in thinking about mathematics. Using games and interactives to replace standard practice exercises contributes to students' understanding as well as their affect toward mathematics. A plethora of rich activities related to number sense topics are offered at Nrich Maths' website (University of Cambridge, n.d.). In middle grades, for example, the Dozens game challenges students to find the largest possible three-digit number that uses two given digits, and one of the player's choosing, and is a multiple of 2, 3, 4, or 6. As students form strategies, they develop a sense for the connections between divisibility and place value in a fun way. In Take Three from Five, students are challenged to find a counterexample set of five whole numbers, which has no subset of three numbers summing to a multiple of three. For high school, the Generating Triples activity challenges students to investigate, then generate, Pythagorean triples.

The foundations of number sense laid in transitional kindergarten through grade five, with an emphasis on counting, ordering place value, and fractions, are built upon in grades six through eight. In turn, as middle-grade students explore rational numbers and the connections between ratios, fractions, decimals, and percents; utilize number lines to compare numbers; engage in proportional reasoning; and generalize numbers and operations to expressions involving variables, they are prepared to understand the high school mathematics in the three critical number sense areas of functions, number systems, and quantitative reasoning.

High School Grades, Nine Through Twelve

The number sense students developed in kindergarten through grade eight culminates in three important areas of learning in the high school grades. First, students see the parallels between numbers (and how they interact) and functions, especially polynomials and rational functions. Second, students extend their understanding of prior number systems, including wholes, integers, and rationals, to learning about the real and complex number systems, which form the basis for algebra and set the stage for calculus. Third, students draw upon their number sense, developed in earlier grades, to cultivate the necessary quantitative reasoning needed to understand and model problems, especially in the area of financial literacy. By complementing an increased understanding of decimals, fractions, and percents with functions, modeling, and prediction, they are equipped to understand financial concepts, tools, and products. Quantitative reasoning is an area that extends well beyond mathematics; quantitative reasoning is defined as the habit of mind to consider both the power and limitations of quantitative evidence in the evaluation, construction, and communication of arguments in public, professional, and personal life (Grawe, 2011).

Three big ideas (Boaler, Munson, and Williams, 2018) related to number sense for the high school level call for students to do the following:

- See parallels between numbers and functions
- Develop an understanding of real and complex number systems
- Develop financial literacy

How Do Students See the Parallels Between Numbers and Functions in Grades Nine Through Twelve?

A deep realization for students to explore in higher math courses is that objects of one type have relationships with each other that parallel the relationships that objects of a different type possess. One of the earliest introductions to this concept of parallelism occurs for students as they compare the behavior of numbers to the behavior of polynomials. In drawing upon their knowledge of integers, specifically as a system of objects with properties, students can see polynomials as an analogous system in terms of the major operations of addition, subtraction, multiplication, and division (A-APR.1). Understanding the parts of a system and how the parts work together in defining the whole system, whether a system of numbers or a system of functions, is another example of CC3 (Taking Wholes Apart and Putting Parts Together).

Moreover, students’ number sense about divisibility concepts, developed in earlier grades while working with integers and rational numbers, can now be extended to explore similar divisibility concepts in the new territories of polynomials and rational functions. Familiar terms such as factors, primes, and fractions take on new meaning for students as they learn to rewrite algebraic expressions by factoring (A-SSE.2) and learn to solving quadratic equations (A-SSE.3.a). The vignette High School Mathematics I/Algebra I: Polynomials Are Like Numbers provides an example of such parallelism in an activity.

How Do Students Develop an Understanding of the Real and Complex Number Systems in Grades Nine Through Twelve?

In high school, algebraic properties and number concepts used in prior grades, such as the distributive property or inverses, are applied in a broader context to explore number systems, especially real and complex numbers. Students’ number sense about rational numbers is critical to understanding the connections between rational number exponents and radical notation (N-RN.1) as well as in rewriting expressions involving radicals and exponents (N-RN.2). For example, students’ ability to perform operations with a fractions representation of rational numbers is needed in shifting forms between equivalent expressions such as $\left(\sqrt{5}\right)^{1/3} = 5^{1/6}$ or $2^{2/3} \cdot 4^{1/2} = 2^{5/3} = (2^5)^{1/3} = (32)^{1/3}$.

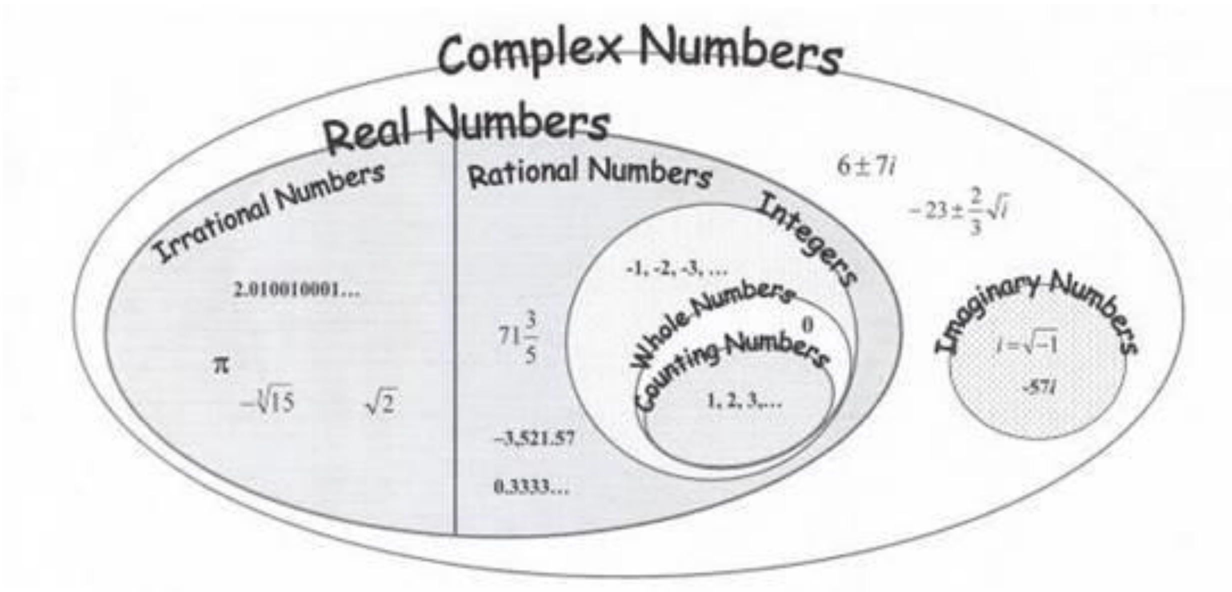
Not only does number sense involving rational numbers inform understanding of exponents and radicals, it also forms the basis for a deep understanding of more advanced topics, such as logarithms and exponential functions. Despite the need, at times, to perform calculations to expand or simplify expressions, students also need to gain proficiency in their reasoning and communication abilities with peer-based conversations on more subtle properties, such as explaining why the sum or product of two rational numbers is rational or discovering that the sum of a rational number and an irrational number is irrational (N-RN.3). It is difficult to overstate the need for students to be comfortable with rational expressions involving irrationals, such as $\sqrt{2}$ and π, as expressions involving these types of numbers are intrinsic to the mathematics present in STEM fields.

The arithmetic skills students have used prior form the basis of their ability to understand operations involving complex numbers. As solving equations increasingly becomes an emphasis in higher math courses, the number systems can begin to be seen as the sets where solutions live. For example, the solutions to linear equations exist entirely in the rational number system. Once students have fully explored this relationship between sets of solutions and sets of numbers, they have the means to understand that solving the simple quadratic equation $x^2 + 1 = 0$ requires a new type of number, i, where $i^2 = -1$. In this manner, students can see that the complex number system, consisting of all numbers of the form a + bi (N-CN.1), provides solutions to polynomial equations, in a similar way to the real system.

This connection between solutions and sets of numbers is extended as students solve quadratic equations with real coefficients (N-CN.3) and discover the three cases that result: a repeated real, two distinct real, or a complex (conjugate) pair of solutions. Students' conception of the complex number system and its itinerant properties grows further with adding, subtracting, and multiplying complex numbers together (N-CN.2), just as they have manipulated prior types of numbers, such as rational numbers, with these same operations.

It is well known that number sense has a strong connection to visual representation. Teachers can facilitate understanding of concepts, especially number systems, by promoting visual representations as a means for understanding. An example of this is shown in figure 3.3, which presents a Venn diagram model of the major number systems used throughout mathematics, which efficiently captures the relationships among the major types of numbers.

Figure 3.3 Venn Diagram Model of the Major Number Systems Used Throughout Mathematics



[Long descripton of figure 3.3]

How Does Number Sense Contribute to Students’ Development of Financial Literacy, Especially in Grades Nine Through Twelve?

Financial literacy is defined as the knowledge, tools, and skills that are essential for effective management of personal fiscal resources and financial well-being. Gaining mathematical knowledge is the first step toward developing financial literacy, which in turn provides early opportunities for meaningful mathematical modeling. The global economic downturn that occurred late in the first decade of the 2000s highlighted the need for increased financial education for school-age students as well as adults. A 2018 survey conducted by the Financial Industry Regulatory Authority (FINRA) showed that only 34 percent of the Americans surveyed had demonstrated basic financial literacy on a short quiz. Alarmingly, the trend over time indicates that financial literacy among Americans is diminishing. Financial education makes a difference, as receiving more than 10 hours of financial education can make a significant difference in an individual’s ability to spend less than they earn (FINRA, 2019).

There are several places in the CA CCSSM that are applicable to financial literacy and number sense. These include standards under the cluster Reason Quantitatively and Use Units to Solve Problems (N-Q.1, N-Q.2, N-Q.3) as well as the standards involving creating and reasoning with equations and inequalities (A-CED and A-REI). By setting contexts in which number sense plays a role in financial decision-making at the high school level, learning can be more authentic. For example, in roughly determining the length of time that a student can realistically save for a large purchase at their current wage rate, a student is using number sense in constructing a simple estimate. In addition, students can use number sense to efficiently compare the ongoing costs associated with a service to a one-time purchase. For example, a student can calculate the difference in purchasing an ongoing gym membership at $40/month versus the one-time purchase cost of $300 for workout equipment to be used at home. The student can include additional factors to help in making their decision, such as the cost per use and amount of time.

Another example that not only relies on number sense but also involves building functions (F-BF.1) is the following:

> Kai arrived at college and was given two credit cards. He didn’t really know much about managing his money, but he did understand how to use the cards—so he bought a few things for his dorm room, including a laptop for $800 and a microwave for $200. Each of the items was purchased with a different credit card, and each card had a different interest rate. The laptop was purchased with a card that had a 15% annual interest rate; the microwave was purchased with a card that had a 25% annual interest rate. At Kai’s job, he earns $1500 per month and spends $1200 per month on school-related and living expenses.
>
> 1. What questions do you have about each credit card that would help you advise Kai on how to pay off each of his debts? (For example, students might ask about the minimum payments required for each card, late charges, and so forth.)

2. If Kai takes the amount of money he has left after paying his other expenses and splits it between the two cards, how long would it take him to pay off each account?
3. What other options does Kai have for paying off the debts?
4. Which option would result in Kai paying the least amount of interest?
    a. Write one or more equations to model the situation and support your answer.
    b. What is the total amount of interest Kai will end up paying for each credit card?

There are two sets of national standards that teachers may use to influence their instruction. The Jump$tart Coalition for Personal Financial Literacy created and maintains the National Standards for Personal Finance Education (Jump$tart and CEE, 2021). These standards describe financial knowledge and skills that students should be able to exhibit. The Jump$tart standards are organized under six major categories of personal finance:

- Spending and Saving: Apply strategies to monitor income and expenses, plan for spending and save for future goals.
- Credit and Debt: Develop strategies to control and manage credit and debt.
- Employment and Income: Use a career plan to develop personal income potential.
- Investing: Implement a diversified investment strategy that is compatible with personal financial goals.
- Risk Management and Insurance: Apply appropriate and cost-effective risk management strategies.
- Financial Decision Making: Apply reliable information and systematic decision making to personal financial decisions.

The second set of national standards available to teachers is the National Standards for Financial Literacy published by the Council for Economic Education (CEE). The CEE standards are available from the Council for Economic Education (Council for Economic

Education, n.d.) and, like the Jump$tart standards, are organized under six major categories of personal finance:

- Earning Income
- Buying Goods and Services
- Saving
- Using Credit
- Financial Investing
- Protecting and Insuring

Although California has not adopted its own standards for financial literacy, the California Council on Economic Education (CCEE) has a number of resources for K–12 teachers (CCEE, n.d.). In addition, the California *History–Social Science Framework* includes language and description of financial literacy as it pertains to global citizenship as well as personal finances (California Department of Education, 2017, 315–316, 559–560).

Conclusion

This chapter presents number sense as a valuable, practical form of intuition and reasoning that a student develops about number. Number sense typically starts to develop naturally, before formal schooling, and continues to develop beyond the school years into adulthood. Interesting and challenging opportunities to reason about and “play” with numbers both in and out of the classroom foster the growth of number sense. When students have number sense, they work with numbers flexibly and choose strategies appropriate to a given problem situation, frequently simplifying the path to a solution. Fluency, an important element of number sense, involves the use of strategies that are flexible, efficient, and accurate. Fluency is developed in partnership with conceptual understanding.

The chapter also highlights the value of math talks and games. Math talks contribute to the development of number sense in every grade. Within each grade band, specific suggestions of topics for math talks are offered, along with websites that present additional ideas. Games, meanwhile, can be used in the classroom to provide students with varied, interesting, and playful exploration and skill practice, as well as to increase students’ positive regard for mathematics.

At every grade, from transitional kindergarten through grade twelve (and beyond), students use number sense to elevate their mathematical capacity. From the early study of place value, arithmetic operations, and fractions in primary grades, to studying rational numbers, number lines, and proportional relationships in the middle grades, to studying functions (including polynomials and work with exponents), building expressions, and financial mathematics applications, the growth of students’ number sense allows for and informs their ability to make sense of problems and to appreciate, rather than fear, all the ways numbers are present in our world.

Long Descriptions for Chapter 3

Figure 3.3: The major number systems used throughout mathematics

Venn diagram that represents the number system. Counting numbers are nested in whole numbers, which are nested in integers, which are nested in rational numbers. The rational numbers and the irrational numbers make up the real numbers, which can be combined with imaginary numbers to make complex numbers. Examples of each type of number are given as well. Return to figure 3.3 graphic